Groups and Lie rings with a Frobenius group of automorphisms

Valentina Iusa

Supervisors

Dr Sandro Mattarei

and

Professor Evgeny Khukhro

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School of Mathematics and Physics
University of Lincoln
United Kingdom

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Declaration

I hereby declare that this thesis contains no material that has been submitted previously, in whole or in part, for the award of any other academic degree or diploma. Except where otherwise indicated, this thesis is my own work.

Valentina Iusa

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Abstract

Suppose that a Frobenius group FH, with kernel F and complement H, acts by automorphisms on a finite group G. Work of E. Khukhro, N. Makarenko and P. Shumyatsky showed that many properties of G are influenced by the corresponding properties of the centraliser of H in G, and possibly by the order of H as well. In particular, they proved that if F is cyclic acting fixed-point-freely and the centraliser of H is nilpotent of class c, then the nilpotency class of G can be bounded from above by a function of c and the order of H.

In this thesis we prove that the dependence of the bound on the order of the Frobenius complement is essential, by constructing an explicit example. We also discuss a generalisation of the quoted result of Khukhro, Makarenko and Shumyatsky to the case of abelian non-cyclic kernels.

These results are obtained by application of different Lie ring methods. In the former case, the desired family of groups is constructed by using the Lazard correspondence. The latter result relies on properties of Lie rings admitting a grading over an abelian group with few non-trivial components and many commuting components.

The final chapter can be read independently of the rest of the thesis. In fact it presents our contribution to a different and independent problem, that is the classification of certain Lie algebras of maximal class.

Contents

Introduction

Let A be a group acting by automorphisms on a group G. Denote by $C_G(A)$ the centraliser of A in G, namely the fixed-point subgroup of A. Experience shows that in many cases the properties of G are influenced by those of $C_G(A)$. After Mazurov's problem 17.72 in Kourovka Notebook [MK10], special attention was given to the case where a Frobenius group acts by automorphisms on a finite group.

Frobenius groups, which play an important role in many areas of Group Theory, were historically introduced by Georg Frobenius in terms of permutation groups. They are transitive groups in which only the identity fixes more than one point and at least one non-trivial permutation fixes one point. According to Frobenius' theorem, as abstract groups, Frobenius groups FH can be characterised as semidirect products of a normal subgroup F, called kernel, by a subgroup H, called complement, such that $C_F(h) = 1$ for every non-trivial element h of H.

The structure of Frobenius groups is quite restricted and well known. In particular, by Thompson's theorem [Tho59] the kernel is nilpotent and, by Higman's theorem [Hig57] its nilpotency class is bounded in terms of the least prime divisor of the order of H. The explicit upper bound is due to Kreknin and Kostrikin (see [Kre63, KK63]).

Suppose that a Frobenius group FH acts on a finite group G in such a way that $C_G(F)$ is trivial. By Belyaev and Hartley's theorem [BH96], the group G is soluble. Moreover, by Khukhro–Makarenko–Shumyatsky's theorem [KMS14, Theorem 2.7] if $C_G(H)$ is nilpotent, then G is nilpotent. In the same article, the three authors proved that if in addition F is cyclic, then the nilpotency class of G can be bounded in terms of the order of H and the nilpotency class of $C_G(H)$ (see [KMS14, Theorem 5.8]). We remark that this result can be seen as part of the more general goal of expressing the properties of G in terms of the corresponding properties of $C_G(H)$, possibly depending also on the order of H (see also [MS10, Shu11, KS11, Khu12]).

The above work left unknown whether the dependence of the bound for the nilpotency class of G on the order of the Frobenius complement H was essential. In this thesis we provide a conclusive answer, by giving the following result.

Theorem. *There exists a family $\mathfrak{G}$ of finite nilpotent groups, of unbounded nilpotency class, whose members G satisfy the conditions:*

1. *G admits a metacyclic Frobenius group of automorphisms;*
2. *the centraliser of the Frobenius kernel in G is trivial;*
3. *the centraliser of the Frobenius complement in G is abelian.*

The above result relies on an analogous one for Lie algebras, that represents the most elaborate part. Its proof will consist of the explicit construction of a family $\mathfrak{L}$ of metabelian Lie algebras satisfying analogous properties. In this case, the transition from Lie rings to finite groups is obtained directly from the Lazard correspondence.

Furthermore, our study extends to the case of abelian non-cyclic Frobenius kernels. Specifically, our result is as follows.

Theorem. *Let G be a finite group admitting a Frobenius group of automorphisms FH, with abelian kernel F and complement H of order q. Suppose that $C_G(H)$ is abelian and F acts coprimely on G in such a manner that*

$C_G(F)$ is trivial. If q^2 is smaller than the least prime divisor of $|F|$, then G is nilpotent of q-bounded class.

For bounding the nilpotency class of the group G, another Lie ring method is used. Indeed, in this case we will consider the associated Lie ring constructed from the lower central series. Specifically, the proof of the theorem above is based on a result about Lie rings admitting a grading over an abelian group with few non-trivial components and many commuting components. We will see that the transition to finite groups is then almost straightforward in case of a coprime action.

We now briefly describe the structure of this thesis. The first chapter is devoted to the preliminary material. After recalling the main definitions, we present some classical results about finite groups admitting a fixed-point-free automorphism. We then look at Frobenius groups as groups of automorphisms and collect general information about the structure of the fixed points of the Frobenius complement.

Chapter 2 is devoted to the description of the Lie ring methods used. All Lie ring techniques provide a recipe for translating some group-theoretic questions to Lie-theoretic ones. The advantage lies in the fact that it is generally easier to deal with Lie rings as they are more linear objects. Nevertheless, both steps, from the group to the Lie ring and back, may be quite non-trivial.

We will consider the associated Lie ring constructed from the lower central series. We will also discuss some problems that may arise when passing from the group to this linearised version of it. We will then show an application of this technique to the study of finite groups with a Frobenius group of automorphisms, by sketching the proof of Theorem 5.8 of [KMS14]. This proof has the advantage of highlighting some key aspects of this kind of problems. It also shows why this method works well in many important cases such as Lie algebras and periodic Lie rings.

Our result about Frobenius groups with abelian non-cyclic Frobenius kernels is proved in Chapter 3. The construction of the example showing the essential dependence of the bound for the nilpotency class on the order of the Frobenius complement is done in Chapter 4.

The final chapter can be read independently of the rest of the thesis, as we consider a completely different problem: the classification of certain Lie algebras of maximal class. The study of graded Lie algebras of maximal class $\bigoplus_{i\geq 1} L_i$ was initiated in [CMN97], discovering a great complexity of structures in positive characteristic. Those Lie algebras were assumed to be generated by their component L_1, a natural assumption which is satisfied if L is the graded Lie algebra associated with the lower central series of a p-group or pro-p group. Subsequent efforts extended to graded Lie algebras of maximal class without that assumption. In particular, graded Lie algebras of maximal class $\bigoplus_{i\geq 1} L_i$ were considered, generated by L_1 and L_n, meaning that those components are one-dimensional and $L_i = 0$ for $1 < i < n$, briefly referred to as having type n.

In his PhD thesis [Sca14] Claudio Scarbolo considered graded Lie algebras of maximal class of type p, where p is an odd prime equal to the characteristic of the underlying field, obtaining a classification under certain assumptions. Crucial role in the classification is played by the first constituent.

Here we introduce the preliminary notions and present our contribution, which consists in a new approach for the determination of the length of the first constituent. This approach offers simpler calculations and a streamlined strategy, obtaining all the required information from a reduced set of relations. Part of our simplification consists in the systematic translation of Lie algebra relations into conditions on univariate polynomials, which allow a more transparent route to the desired conclusions.

Chapter 1

Preliminaries

This chapter sets the scene for this research. In the following sections we collect all the preliminary material: definitions, well-known facts and classical theorems. In our opinion all of them, even if not specifically mentioned, contribute strongly to the framework and enable an understanding of the topic.

In the first section we recall basic definitions and set the notation. The second section is devoted to Frobenius groups and their structure. Section 1.3 provides selected literature about regular automorphisms of finite groups. This leads us to the last section, where Frobenius groups are considered in their action on finite groups and Lie rings. First, we present an overview of what is already known, the open problems and the recent developments in this research topic. The final subsection is more technical, collecting general information about the structure of the fixed points of the Frobenius complement.

1.1 Preliminary definitions

To begin, we recall some basic definitions and set the notation. In this thesis all groups are going to be finite. For subsets M and N of a group we write

$$M \cdot N = \{m \cdot n \,|\, m \in M,\, n \in N\}\,.$$

By $G = B \rtimes A$ we denote a semidirect product of B and A, that is

$$G = B \cdot A, \qquad A \leq G,\, B \trianglelefteq G \qquad \text{and} \qquad B \cap A = 1.$$

For simplicity, we might use the notation BA in cases where it has been already made clear that we are dealing with a semidirect product. A subgroup of a group is said to be characteristic if it is invariant under all automorphisms of the group.

By $[g, h] = g^{-1}h^{-1}gh$ we denote the commutator of the elements g and h in a group. The mutual commutator subgroup of arbitrary subsets M and N of a group is defined by

$$[M, N] = \langle [m, n] \,|\, m \in M,\, n \in N\rangle.$$

The derived subgroup of a group G is denoted by $G' = [G, G]$. The terms of the derived series of G are defined by induction

$$G^{(0)} = G, \qquad\qquad G^{(i)} = [G^{(i-1)}, G^{(i-1)}].$$

We say that the group is soluble if there exists an integer d such that $G^{(d)} = 1$. If G is soluble, then the derived length of G is the smallest integer satisfying this property. The derived words are defined recursively by

$$\delta_0(x_1) = x_1, \qquad \delta_i(x_1, \ldots, x_{2^i}) = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})].$$

The terms of the lower central series are defined by

$$\gamma_1(G) = G, \qquad\qquad \gamma_i(G) = [\gamma_{i-1}(G), G].$$

We say that G is nilpotent if there exists an integer c such that $\gamma_{c+1}(G) = 1$. If G is nilpotent, then the nilpotency class is defined as the smallest integer satisfying this property.

We define composite commutators in the elements of a set X (as formal expressions) inductively by their weight. The elements of X are the commutators of weight 1; if c_1 and c_2 are commutators of weight respectively w_1 and w_2 in elements of X, then $[c_1, c_2]$ is a commutator in elements of X of weight $w_1 + w_2$. The commutator $[\ldots[[x_1, x_2], x_3], \ldots, x_k]$ is called simple and is denoted simply by $[x_1, x_2, \ldots, x_k]$.

A group G is said to have finite exponent (or period) n if $g^n = 1$ for all $g \in G$. The minimal number of generators of a finite abelian group is called the rank. An abelian group of prime exponent p is called elementary. It can be regarded as a vector space over the finite field F_p of p elements. The automorphisms of the group are the bijective linear transformations of this vector space.

A chain of nested subgroups

$$1 \leq H_1 \leq H_2 \leq \ldots \leq H_n = G$$

is said to be subnormal if $H_i \trianglelefteq H_{i+1}$ for all i. It is said to be normal if $H_i \trianglelefteq G$ for all i. In both cases the factor-groups H_{i+1}/H_i are called factors of the series.

Although, as a rule, the group operation is denoted by the multiplication sign (often omitted) and the identity element by 1, in the case of abelian groups sometimes additive notation is used.

Let π be a set of prime numbers. Its complement in the set of all primes is denote by π'. A subgroup of a finite group G whose order is divisible only by prime numbers from π and its index only by primes from π' is called Hall π-subgroup of G.

A module over a commutative ring K with identity, called also a K-module, is an additive group M which admits multiplication by the elements of K

satisfying the following axioms

$$\begin{aligned}
\alpha(x+y) &= \alpha x + \alpha y, && \text{for all } \alpha \in K,\ x, y \in M\\
(\alpha+\beta)x &= \alpha x + \beta x && \text{for all } \alpha, \beta \in K,\ x \in M;\\
(\alpha\beta)x &= \alpha(\beta x) && \text{for all } \alpha, \beta \in K,\ x \in M;\\
1x &= x && \text{for all } x \in M.
\end{aligned}$$

Thus modules over a field are precisely the vector spaces. Every abelian group may be regarded as a $\mathbb{Z}$-module in a natural way:

$$na = \underbrace{a + \cdots + a}_{n} \qquad (-n)a = n(-a) \qquad \text{and} \qquad 0a = 0.$$

The elements $m_1, \ldots, m_s$ are said to generate the K-module M if every element m in M can be written in the form $m = \sum_{i=1}^{s} k_i m_i$, where $k_i \in K$.

If G is a group of linear transformations of a vector space V over a field K, then V may be regarded as a KG-module, where KG is the group ring. In an analogous way if G is a group of automorphisms of an abelian group V, then V may be regarded as $\mathbb{Z}G$-module.

Let M and N be K-modules. Their tensor product $M \otimes_K N$ is defined as the factor-module of the free K-module with free generators $m \otimes n$, for $m \in M$ and $n \in N$, by the submodule generated by all elements of the form

$$\begin{aligned}
k(m \otimes n) - km \otimes n, \qquad km \otimes n - m \otimes kn,\\
m \otimes (n_1 + n_2) - (m \otimes n_1 + m \otimes n_2),\\
(m_1 + m_2) \otimes n - (m_1 \otimes n + m_2 \otimes n),
\end{aligned}$$

where $k \in K$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$.

If the K-modules $M = \oplus_i M_i$ and $N = \oplus_j N_j$ are decomposable into direct sums of K-submodules, then their tensor product is decomposable into direct sum of K-submodules as well: $M \otimes_K N = \bigoplus_{i,j} M_i \otimes_K N_j$.

In this thesis we will use tensor products to extend the ground ring of a module.

A Lie ring L is an abelian group equipped with a non-associative multiplication, usually denoted by brackets $[\cdot,\cdot]$, satisfying the following axioms:

$$
\begin{aligned}
[x,x] &= 0 \qquad \text{(anticommutativity)}\\
[[x,y],z]+[[z,x],y]+[[y,z],x] &= 0 \qquad \text{(Jacobi identity)},
\end{aligned}
$$

for every $x,y,z \in L$. The identity $[x,y]=-[y,x]$ follows immediately from anticommutativity. For Lie rings the notion of left, right and two-sided ideals coincide. If I and J are ideals of a Lie ring L, then the additive subgroup generated by the set of commutators $\{[i,j]\,|\,i \in I, j \in J\}$ is an ideal of L, denoted by $[I,J]$.

Commutators in elements of a subset X of a Lie ring L and their weights are defined and denoted exactly in the same way as group commutators.

The ideal of L generated by a set X is denoted by ${}_{id}\langle X\rangle$. Its additive group is generated by the simple Lie brackets $[x,l_1,\dots,l_k]$, with $x \in X$ and $l_i \in L$. The subring generated by the set X is denoted by $\langle X\rangle$.

The derived subring of a ring L is denote by $L'=[L,L]$. The terms of the derived series of L are defined inductively:

$$
L^{(0)} = L, \qquad\qquad L^{(i)} = [L^{(i-1)}, L^{(i-1)}].
$$

The terms of the lower central series are also defined by induction:

$$
\gamma_1(L) = L, \qquad\qquad \gamma_i(L) = [\gamma_{i-1}(L), L].
$$

Soluble Lie rings, nilpotent Lie rings, derived length and nilpotency class are defined exactly as for groups.

If a Lie ring is also a K-module, then L is said to be a K-Lie algebra. If K is a subring of R, then the R-module $R \otimes_K L$ may be regarded in a natural way as an R-Lie algebra, with multiplication given by

$$
[r_1 \otimes l_1, r_2 \otimes l_2] = r_1 r_2 \otimes [l_1, l_2].
$$

It follows easily from the definition that

$$(R \otimes_K L)^{(i)} = R \otimes_K L^{(i)}, \qquad \gamma_i(R \otimes_K L) = R \otimes_K \gamma_i(L).$$

In particular, the derived length and the nilpotency class of $R \otimes_K L$ equal respectively the derived length and the nilpotency class of L.

Let G be an abelian group. A Lie ring L is said to have a G-grading (or, equivalently, to be G-graded) if to each element $g \in G$, there corresponds a subgroup L_g of the additive group of L such that

$$L = \bigoplus_{g \in G} L_g \quad \text{and} \quad [L_g, L_h] \subseteq L_{g+h} \quad \text{for all } g, h \in G.$$

An important example of grading is given by the eigenspace decomposition. For instance, if L is a $\mathbb{C}$-Lie algebra and ϕ is an automorphism of L of finite order n, then L decomposes into the direct sum of the eigenspaces of ϕ:

$$L = \bigoplus_{i=0}^{n-1} L_i,$$

where $L_i = \left\{ l \in L \,|\, l^{\phi} = \omega^i l \right\}$ and ω is a primitive nth root of unity. This decomposition gives a $(\mathbb{Z}/n\mathbb{Z})$-grading of L.

An ideal I of a G-graded Lie ring $L = \bigoplus_{g \in G} L_g$ is said to be homogeneous if $I = \bigoplus_{g \in G} I \cap L_g$.

1.2 Frobenius groups

In this section we recall the definition of a Frobenius group and its most important properties. Here theorems are stated without proofs. The interested reader will find expositions in the books [Isa08, Chapter 6] and [Hup67, Chapter 5.8]. We also cite Passman's book [Pas68], where Frobenius groups are presented as permutation groups. Indeed, they are transitive permutation groups on finite sets such that no non-trivial element fixes more than one point and some non-trivial element fixes a point.

A Frobenius group G can also be defined as a finite group containing a proper non-trivial subgroup H such that

$$H^g \cap H = \{1\}, \qquad \text{for all } g \in G \setminus H. \tag{1.1}$$

Such a subgroup is called a Frobenius complement.

In 1901, Frobenius proved an unexpectedly strong amount of structure on Frobenius groups.

Theorem 1.1 (Frobenius' Theorem). *Let G be a Frobenius group with complement H. There exists a normal subgroup F of G, called the* Frobenius kernel*, such that G is a semidirect product of F by H. The conjugation action of each non-trivial element h of H is fixed-point-free, that is*

$$C_F(h) = \left\{ f \in F \ : \ f^h = f \right\} = 1.$$

Conversely, any semidirect product of a normal subgroup F by a subgroup H such that $C_F(h) = 1$ for every $1 \neq h \in H$ is a Frobenius group.

A few illustrative examples are given below.

Example 1.2. For every finite field F_q, where $q > 2$, the group of the invertible affine transformations $\{\phi_{a,b} : x \mapsto ax + b \,|\, a, b \in F_q,\ a \neq 0\}$ of F_q is a Frobenius group. In this case the kernel is given by the subgroup of translations $\{\phi_{1,b}\} \cong F_q$. The complement consists of the rotations $\{\phi_{a,0}\} \cong F_q^*$.

Example 1.3. The dihedral group D_{2n} is a Frobenius group whenever n is odd. The kernel is given by the rotations and a complement can be taken as $\langle t \rangle = \{1, t\}$, for any reflection t.

More generally, when the Frobenius group is considered as a permutation group on a finite set X, the Frobenius kernel is given by the maps with no fixed points together with the identity. A Frobenius complement consists of a point stabiliser, for any point chosen in X.

In the next example we see the case of a Frobenius group with a nilpotent non-abelian kernel.

Example 1.4. Let F be the group of order 7^3 consisting of

$$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in F_7 \right\}$$

and let the map $\phi : F \longrightarrow F$ act on F in the following way

$$\phi \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2x & 4y \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not hard to prove that ϕ is an automorphism of order 3 and no fixed points. By Frobenius' theorem, we can conclude that the semidirect product $F \rtimes \langle \phi \rangle$ is a Frobenius group.

It is worth noting that the definition given by Property (1.1) makes perfect sense in the infinite case as well. However, the structure theorem stated above does not carry over to infinite Frobenius groups. One can refer to [DM96] and [Col90] for several examples showing how badly it fails. Essentially, the object in Frobenius' theorem that plays the role of the kernel, that is

$$F = \left(G \setminus \bigcup_{g \in G} H^g \right) \cup \{1\},$$

is not even a subgroup in the infinite case.

In this thesis we will always consider finite Frobenius groups.

The characterisation of (finite) Frobenius groups as a semidirect product raises questions about the structure of Frobenius kernels and Frobenius complements. The main result on Frobenius complements is due to the classical work of W. Burnside and H. Zassenhaus, who determined the structure of all finite fixed-point-free automorphism groups. It can be summarised as follows (see [Hup67, Chapter 5, Theorems 8.15 – 8.18]).

Theorem 1.5. *Let G be a Frobenius group with kernel F and complement H. Then the following hold:*

1) All Sylow subgroups of H are cyclic or generalised quaternion.

2) If H has even order, then it contains a unique element h of order 2. The kernel is abelian and $f^h = f^{-1}$, for all f in F.

3) If H has odd order, then it is metacyclic. If $h \in H$ has prime order, then the subgroup generated by h is normal and either contained in the center $Z(H)$ or in the derived subgroup H'.

Concerning the kernel, it became important to understand which groups admit a fixed-point-free automorphism of prime order. The long-standing conjecture about the nilpotency of such groups was finally proved by Thompson in 1959.

Theorem 1.6 (Thompson [Tho59])**.** *A finite group that admits a fixed-point-free automorphism of prime order is nilpotent.*

Therefore, finite Frobenius groups have a quite restricted structure. Here we list some other well-known properties of these groups. Some of them follow directly from the theorems above, others require different arguments. We refer the reader to [Hup67] for their proofs.

1. The Frobenius kernel is unique and the Frobenius complement is uniquely determined up to conjugacy.

2. $|H| | (|F| - 1)$, hence the action of H on F is coprime.

3. All abelian subgroups of H are cyclic.

4. If F is cyclic, then H is cyclic too.

Information about irreducible representations of a Frobenius group can be found in [Isa76] and [Hup98] and can be summarised as follows.

Theorem 1.7. *Let $G = FH$ be a Frobenius group, with kernel F and complement H. Suppose that V is an irreducible KG-module, where K is a field of characteristic not dividing $|F|$. We have one of the following*

1. *either F is in the kernel of the representation and V is an irreducible H-module;*

2. *or V is induced from an irreducible F-module.*

The theorem above will be enough for our purposes. The interested reader is referred to Isaac's book for the case where the characteristic of the field divides the order of the Frobenius kernel.

1.3 Groups and Lie rings with a fixed-point-free automorphism

As essential background of our research, we study the fixed-point-free (also called regular) automorphisms of a finite group. The literature on this subject is vast, but to keep this section relatively short, we collect only results which will be useful later on. In particular, all theorems stated here investigate the degree of commutativity of such groups.

Even before Thompson proved Theorem 1.6 about the nilpotency of a group with a fixed-point-free automorphism of prime order, G. Higman had already asked whether one could say anything about its class in case it is nilpotent. Indeed, in the late 1950's, two fairly simple results were well-known: a group with a fixed-point-free automorphism of order 2 is abelian and a group with a fixed-point-free automorphism of order 3 has nilpotency class at most 2 (Burnside, [Bur97]). This led Higman to ask whether (nilpotent) groups with a fixed-point-free automorphism of prime order p must be of nilpotency class at most some number $h(p)$.

In 1957, Higman proved that such a number does exist.

Theorem 1.8 (Higman [Hig57]). *The nilpotency class of a finite group admitting a fixed-point-free automorphism of prime order p is bounded from above by a function of p only.*

As a matter of fact, because Higman was working before Thompson's proof, he had to make the additional assumption that the group in question was soluble. The function $h(p)$ is now called Higman's function in his honour.

Despite the fact that the proof did not provide any explicit formula for $h(p)$, it is of great importance because, there, Lie ring methods are applied for the first time to the study of group automorphisms (see Chapter 2). In fact, Theorem 1.8 holds for Lie rings as well and the bound would be given by the same Higman's function.

Furthermore, in the same article, Higman proved that

$$h(p) \geq \frac{p^2 - 1}{4}. \tag{1.2}$$

This lower bound is achieved by producing examples. Essentially, Higman gives a Lie ring by generators and relations and proves that it has nilpotency class at least $(p^2-1)/4$. Then, for every prime p' exceeding this number, one can construct a p'-group with the same nilpotency class by using the Lazard correspondence.

The current conjecture is that the inequality (1.2) is sharp. However, this has been proved only for very small values of p so far. The results for $p = 2$ and $p = 3$ are folklore. Proofs may be found in [Gor68, Chapter 10]. The fact that $h(5) = 6$ was proved in [Hig57]. Two independent works showed that $h(7) = 12$. Scimemi's proof, based on computer calculations, was never published. For a computer-free argument see [Hug85] instead.

Further steps were made by Kreknin and Kostrikin. They gave an explicit upper bound for the function $h(p)$. Their result was originally for Lie rings and consisted of two parts. First, they bounded the derived length of a

Lie ring with a fixed-point-free automorphism of finite (not necessary prime) order.

Theorem 1.9 (Kreknin [Kre63]). *If a Lie ring admits a fixed-point-free automorphism of finite order n, then it is soluble and its derived length is not greater than $2^n - 2$. This bound can be reduced by half if n is a prime number.*

Subsequently, assuming a given derived length and the order of the automorphism being a prime, they bounded the nilpotency class of such Lie ring.

Theorem 1.10 (Kreknin, Kostrikin [KK63]). *If a soluble Lie ring of derived length s admits a fixed-point-free automorphism of prime order p, then it is nilpotent and its nilpotency class is at most*

$$\frac{(p-1)^s - 1}{p-2}.$$

This final result can be stated also in terms of finite groups, essentially because the nilpotency class is preserved by passing from a group to its associated Lie ring. The details of this fact will be extensively discussed in Chapter 2. By putting information together we get

$$h(p) \leq \frac{(p-1)^{2^{p-1}-1} - 1}{p-2}. \tag{1.3}$$

It is worth noting that Theorem 1.9 does not imply that a finite group with a fixed-point-free automorphism of finite order is soluble. Indeed, there is no clear way to derive it from the result about Lie rings. Nevertheless, the solubility of such group is a consequence of the classification of finite simple groups (see [Row95]).

The idea of bounding the nilpotency class in terms of the derived length is originally due to P. Hall and his famous criterion for nilpotency stated below (see [Khu93, Theorem 2.3.1] for the proof).

Theorem 1.11 (Hall, [Hal58]). *Let N be a normal subgroup of a group G. If N is nilpotent of class k and $G/[N,N]$ is nilpotent of class c, then G itself is nilpotent and its nilpotency class is bounded by $f(c,k) = (c-1)\dfrac{k(k+1)}{2} + k$.*

The statement also holds for Lie rings, essentially by using the same argument. The induction on the derived length, which is the basis of Kreknin-Kostrikin's result, reverses the more natural procedure of using the nilpotency class to bound the derived length. This might suggest that the bound achieved is unnecessarily large. Moreover, the bound for the derived length provided by Theorem 1.9 looks quite big as well, due to the combinatorial nature of its proof. Indeed, if the conjecture about the sharpness of Equation (1.2) was true, then the derived length would be logarithmic in p.

Many authors worked in order to reduce these bounds. We cite the work of P. Shumyatsky, A. Tamarozzi and L. Wilson [STW05] and a subsequent paper of the last author [Wil07] which provide the best known so far. Unfortunately, they are still exponential in p.

Before concluding this section we state two results, which extend Theorem 1.9 and Theorem 1.10 as special cases. We recall that an automorphism ϕ of a Lie ring L is called semisimple if, after a suitable extension of the ground ring, the additive group of L decomposes into the direct sum of analogues of eigenspaces of the linear transformation ϕ.

Theorem 1.12 (Shalev, [Sha93]). *Let L be a Lie ring admitting a semisimple fixed-point-free automorphism of order n and d distinct eigenvalues. Then L is soluble of derived length $dl(L) \leq 2^{d-1} - 1$.*

An equivalent version of this theorem can be stated in terms of graded Lie rings.

Theorem 1.13. *Let L be an S-graded Lie ring, where S is a finite set of complex numbers not including 1. Then L is soluble. Moreover, if $|S| = d$, then $dl(L) \leq 2^{d-1} - 1$.*

Also the nilpotency class can be bounded in terms of the number of distinct eigenvalues.

Theorem 1.14 (Khukhro, [Khu02]). *Suppose that a Lie ring L admits a semisimple fixed-point-free automorphism of prime order having exactly d distinct eigenvalues. Then L is nilpotent of class at most $\frac{d^s-1}{d-1}$, where $s = 2^{d-1} - 1$.*

Observe that the formula above corresponds to the one given in Theorem 1.10, where d has taken the place of $p-1$.

Before concluding we just mention a natural generalisation of these results. The case of an almost fixed-point-free automorphism has also been considered by Khukhro in [Khu92]. The "almost regularity" of the automorphism implies the "almost nilpotency" of the Lie ring.

Theorem 1.15. *Let ϕ be an automorphism of prime order p of a Lie ring L. If the number of its fixed points is equal to q, then L has a subring of (p, q)-bounded index which is nilpotent of p-bounded class.*

The same result has been proved for groups, modulo the classification of finite simple groups.

1.4 Frobenius groups of automorphisms

In this section we consider Frobenius groups in their action on finite groups. In the first subsection we give an overview of the state-of-the-art for this problem. We summarise the known results as well as some open problems. In Section 1.4.2 we restrict ourselves to the case of a fixed-point-free action of the Frobenius kernel and study the fixed points of the Frobenius complement.

1.4.1 Literature review

Suppose that A is a group acting on a group G. Experience shows that in many cases the properties of G are influenced by those of its subgroup of fixed points

$$C_G(A) = \{g \in G \,:\, g^a = g \text{ for all } a \in A\}\,,$$

called centraliser of A in G. Indeed many authors, among which Turull ([Tur84]) and Berger ([Ber73]), successfully used representation-theoretic methods to restrict the structure of G by using restrictions on the structure of $C_G(A)$.

After Mazurov's problem 17.72 in the Kourovka Notebook [MK10], special attention was given to the case where A is a Frobenius group. In this circumstance, the "additional" action of the Frobenius complement suggests to study the properties of G by looking at those of its centraliser $C_G(H)$.

Indeed, we can look at the simplest case where $G = V$ is a vector space and a Frobenius group $FH \leq \mathrm{GL}(V)$ acts in such a way that $C_V(F) = 0$. Then, by Clifford's theorem, V is a free H-module and $C_V(H)$ is given by the diagonal elements so that $\dim V = |H| \dim C_V(H)$. The belief is that the properties of G are close to those of $C_G(H)$ in more general cases as well.

Strictly speaking, Mazurov focused his attention on the case of double Frobenius groups, that is Frobenius groups FH acting on a group G in such a way that GF is a Frobenius group. He asked whether

1. the nilpotency class of G can be bounded in terms of $|H|$ and the nilpotency class of $C_G(H)$;

2. the exponent of G can be bounded in terms of $|H|$ and the exponent of $C_G(H)$.

Note that, in the first question, the nilpotency of G follows directly from Thompson's theorem.

Big achievements in this direction were made by E. Khukhro, N. Makarenko and P. Shumyatsky (see [Khu08] and [MS10]). In particular, in the latter article the authors solved Mazurov's question 1 in the affirmative, under the extra condition of a coprime action of H.

The same authors, in their articles [Khu12, KMS14], consider a more general situation by dropping the hypothesis of double Frobenius groups. Suppose now that G is a finite group admitting a Frobenius group of automorphisms FH, with kernel F and complement H, such that $C_G(F) = 1$. In this new setting, they derived the following properties of G from the corresponding properties of $C_G(H)$:

1. the order of G equals $|C_G(H)|^{|H|}$;

2. the rank of G satisfies $r(G) \leq |H|\, r(C_G(H))$;

3. the Fitting height of G equals the Fitting height of $C_G(H)$;

4. in the case F is cyclic, the exponent of G is bounded in terms of the exponent of $C_G(H)$ and $|FH|$;

5. if $C_G(H)$ is nilpotent, then G is nilpotent;

6. if in addition F is cyclic, then the nilpotency class of G is bounded in terms of the nilpotency class of $C_G(H)$ and $|H|$.

It is worth noting that here the order of G is not assumed to be coprime to the order of FH. Moreover, bounds in 4 and 6 are very big but can be made explicit.

Clifford's theorem and representation-theoretic techniques are used to show 1, 2, 3 and 5. Results in 4 and 6 are obtained by using two different Lie ring methods. In the former case the authors work with the Lazard's Lie algebra associated with the Jenning-Zassenhaus filtration of a finite p-group.

In the latter, the associated Lie ring is obtained from the lower central series of G.

It is presently unclear, even in the case where GFH is a double Frobenius group, if the bound for the exponent of G can be made independent of the order of F. This would give an affirmative answer to part (2) of Mazurov's problem. Moreover, there is no evidence that the hypothesis about the cyclicity of F is essential.

Despite the fact that the nilpotency of G follows from that of $C_G(H)$, the additional condition about a cyclic kernel is generally necessary for bounding its nilpotency class. This is proved in the same article by providing explicit examples. It is worth mentioning that the result in 6 comes from an analogous one for Lie rings. Nevertheless, in this case the nilpotency of $C_L(H)$ alone does not imply the nilpotency of L.

In Chapter 3 we will extend the bound for the nilpotency class of G to the case of abelian kernels, under the assumption that H is small compared to F in an appropriate way. In Chapter 4 we will prove that the bound depends on $|H|$ in an essential way.

However, many open problems still remain. For instance, the solubility of G is guaranteed by a theorem of Belyaev and Hartley (Theorem 1.19 stated below) but it is not known whether its derived length can be bounded in terms of the derived length of $C_G(H)$, possibly depending on $|H|$ as well. The same problem is open for double Frobenius groups and Lie rings. Other unsolved questions are mentioned in [Khub, Khua].

A natural generalisation was to consider finite groups G with a Frobenius group of automorphisms FH in which the kernel F no longer acts fixed-point-freely but has a relatively small number of fixed points ([Khu13], [KM13b], [KM13a], [EG18]). "Almost fixed-point-free" action of F implies that G is "almost" as good as when F acts fixed-point-freely. Indeed, in the articles mentioned above it is proved that

1. the order $|G| \leq |C_G(F)| \cdot f(|H|, C_G(H))$, for some function f;

2. the rank $r(G) \leq r(C_G(F)) + g(|H|, r(C_G(H)))$, for some function g.

3. if G is soluble, $(|G|, |FH|) = 1$ and $C_G(H)$ has Fitting height f, then the index $[G : F_f(G)]$ is bounded in terms of $|F|$ and $|C_G(F)|$. Here $F_f(G)$ represents the fth term of the upper Fitting series;

4. if F is cyclic, $(|G|, |FH|) = 1$ and $C_G(H)$ is nilpotent of class c, then G has a nilpotent characteristic subgroup whose index is bounded in terms of $|F|$, $|C_G(F)|$ and c. Moreover, its nilpotency class is bounded in terms of c and $|H|$ only.

Finally, we mention the recent work of G. Ercan, İ. Güloğlu and E. Khukhro, who extended some of those results to Frobenius-like groups of automorphisms ([EGK14a, EGK14b]). As the name suggests, Frobenius-like groups generalise Frobenius groups. They have a non-trivial nilpotent normal subgroup F called kernel acted on by a non-trivial complement H via automorphisms so that $[F, h] = F$, for all non-trivial $h \in H$.

1.4.2 Fixed points of Frobenius complements

In this section we discuss the question of covering the fixed points of an automorphism of a group G in any invariant quotient of the group by its fixed points in G. We will also consider the same question for a Frobenius group of automorphims FH, with kernel F and complement H, under the assumption that $C_G(F) = 1$. This is particularly important for the Lie ring method described in the following chapter. Indeed, it would ensure that every commutator identity holding in $C_G(H)$ is also true in $C_{L(G)}(H)$, where $L(G)$ is the Lie ring associated to G. Results collected here can be found in [Khu93] and [Gor68].

First we recall few well-known facts about fixed points of a group automorphism.

Theorem 1.16. *Let G be a finite group, ϕ an automorphism of G and N a normal and ϕ-invariant subgroup of G. Then $|C_{G/N}(\phi)| \leq |C_G(\phi)|$.*

This result may be strengthened in the case where the action is coprime.

Theorem 1.17. *Let G be a finite group, ϕ an automorphism of G and N a normal and ϕ-invariant subgroup of G whose order is coprime to the order of ϕ, that is $(|N|, |\phi|) = 1$. Then*

$$C_{G/N}(\phi) = C_G(\phi)N/N.$$

A straightforward consequence of this theorem is the following result.

Corollary 1.18. *Let G be a finite group and ϕ an automorphism of G such that $(|G|, |\phi|) = 1$. If ϕ centralises all factors of some subnormal series of G, then $\phi = 1$.*

We now discuss the case of an automorphism group. Theorem 1.17 holds in this more general case as well (see [Gor68, Theorem 6.6.2]). However, dropping the assumption of a coprime action, if A is a group acting on a group G and N is a normal A-invariant subgroup of G, then the inclusion

$$C_G(A)N/N \subseteq C_{G/N}(A) \tag{1.4}$$

may not be sharp. In the next theorem, we consider the case where $A = FH$ is a Frobenius group with a fixed-point-free kernel F. It is worth noting that, in this case, G is soluble from the outset, by the following result of Belyaev and Hartley based on the classification of finite simple groups.

Theorem 1.19 (Belyaev, Hartley [BH96])**.** *Suppose that a finite group G admits a nilpotent group of automorphism F such that $C_G(F) = 1$. Then G is soluble.*

Suppose now that N is a normal FH-invariant subgroup of G. The next lemma ensures that the action induced by F on the quotient group G/N is automatically fixed-point-free.

Lemma 1.20. *Let G be a finite group admitting a nilpotent group of automorphisms F such that $C_G(F) = 1$. If N is a normal F-invariant subgroup of G, then $C_{G/N}(F) = 1$.*

The next result establishes that if we consider the Frobenius complement, then inclusion (1.4) is sharp.

Theorem 1.21. *Suppose that a finite group G admits a Frobenius group of automorphisms FH, with kernel F and complement H. If N is a FH-invariant normal subgroup of G such that $C_N(F) = 1$, then*

$$C_{G/N}(H) = C_G(H)N/N.$$

This is obtained by providing a reduction to the case where $(|N|, |F|) = 1$ and by showing that every factor of an unrefinable FH-invariant normal series of G is a free F_pH-module, for a suitable prime p. Another consequence of this fact is the following lemma.

Lemma 1.22. *Suppose that a finite group G admits a Frobenius group of automorphisms FH, with kernel F and complement H, such that $C_G(F) = 1$. Then we have $G = \langle C_G(H)^f \mid f \in F \rangle$.*

For the complete proofs of the last propositions regarding Frobenius groups we refer the reader to [KMS14, Khu10].

Chapter 2

Linear Methods for Nilpotent Groups

The association of Lie algebras with groups originated in the 19th century for Lie groups. The idea of constructing Lie rings from abstract groups arose some time later by Magnus in the context of the restricted Burnside problem. Since then, and due to the outstanding contribution of E. Zelmanov, the applications of Lie rings in Group Theory has been considerably enlarged. Indeed, this construction has proved to be one of the most useful and effective tools in the study of some group identities, the coclass theory of p-groups and pro-p groups and also in the investigation of the fixed points of group automorphisms (see for instance [Mag50, WZ92, Zel90, Sha00]).

This subject includes various techniques which can look quite different from each other. In all cases the Lie ring method provides a recipe for translating some group-theoretic questions to Lie-theoretic ones. This translation is straightforward in some cases, and quite sophisticated in others. For instance, we cannot avoid mentioning the Lazard correspondence based on the Baker-Campbell-Hausdorff formula, as it will be used to produce some examples.

It establishes a correspondence between finite p-groups of nilpotency class less than p and finite nilpotent Lie rings of p-power order and nilpotency class less than p (see for instance [Khu98, Chapter 10.2]). The restriction on the nilpotency class, however, is quite severe for this correspondence to be broadly useful.

Here we focus on the construction of the associated Lie ring from the lower central series of a group (Section 2.1). This method is less precise but much more generally applicable. The purpose of this chapter is to describe the use of this tool in the study of fixed-point-free group automorphisms. This is done in Section 2.3, where we present an application to finite groups with a Frobenius group of automorphisms. When commutation is replaced by a bilinear form, a simplification often comes from operations of the linear structure which have no correspondence in the group. The one most successfully used, and described here in Section 2.2, is extension of the ground-ring.

For an in-depth study of these topics, we recommend the survey [Shu98], and the textbooks [Khu93] [Khu98], [HB82], [VL90].

2.1 The associated Lie ring

This section is devoted to the classical Lie ring construction. The definition of the associated Lie ring is presented in the first subsection, together with its most important properties. In the last subsection we analyse some problems that may arise when replacing a group with this "linearised" version.

2.1.1 Definition and properties of $L(G)$

Among many standard results about the lower central series and commutators of a group, here we state only those which motivate the definition of the associated Lie ring. We refer the reader to [HJ60, Chapter 10] for their proofs.

Proposition 2.1. *Let G be a group and let $\{\gamma_i(G)\}_{i\geq 1}$ be the terms of its lower central series. Let $x, y, z \in G$. Then*

1. $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$;

2. $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$;

3. $[x, y]^{-1} = [y, x]$

4. $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$;

5. $\gamma_i(G)$ *is a fully invariant subgroup of* G *for all* i;

6. $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.

When there is no possibility of confusion, the terms of the lower central series of the group G will be denoted simply by $\{\gamma_i\}_{i\geq 1}$. The factors γ_i/γ_{i+1} are abelian groups for which we use the additive notation, that is

$$\overline{x} + \overline{y} = \overline{xy},$$

for every $x, y \in \gamma_i$. They can be naturally considered as $\mathbb{Z}$-modules.

Definition 2.2. Let G be a group and $\{\gamma_i\}_{i\geq 1}$ its lower central series. The *associated Lie ring* of G, denoted by $L(G)$, has additive group given by

$$L(G) = \bigoplus_{i\geq 1} \gamma_i/\gamma_{i+1}.$$

The factor group γ_i/γ_{i+1} is called the *homogeneous component of weight* i and its elements are called homogeneous. The Lie product in $L(G)$ is induced by taking commutators in G. It is first defined for homogeneous elements by setting

$$[x_i + \gamma_{i+1}, x_j + \gamma_{j+1}] = [x_i, x_j] + \gamma_{i+j+1},$$

for every $x_i \in \gamma_i$ and $x_j \in \gamma_j$. It is then extended to arbitrary elements by using bi-additivity.

Identity 2.1.6, together with 2.1.1 and 2.1.2, ensures that the Lie bracket is well defined. Identity 2.1.4, known as Witt's identity, is the group theoretic analogue of the Jacobi identity. We use square brackets to denote both commutators in the group and Lie products in its associated Lie ring, as the meaning will always be clear from the context.

The Lie ring $L(G)$ has a natural $\mathbb{Z}$-grading, if we put $L_i = 0$ for every $i \leq 0$ and $L_i = \gamma_i/\gamma_{i+1}$ for $i \geq 1$. This follows directly from the definition.

Observe that the associated Lie ring is defined for every group G. However, it only reflects the properties of the quotient of G by its nilpotent residual $N = \bigcap_{i \geq 1} \gamma_i(G)$. Indeed, it is easy to see that $L(G) \cong L(G/N)$. This is why classically G is assumed to be residually nilpotent.

Some important and well-known properties of the associated Lie ring are listed below.

Theorem 2.3 ([Khu98], Theorem 3.2.2)**.** *Let G be a group and let $L(G)$ be its associated Lie ring.*

1. *If G is soluble of derived length d, then $L(G)$ is soluble of derived length not exceeding d.*

2. *If G is nilpotent, then $L(G)$ is nilpotent as well, with class equal to the that of G. Moreover, if G is also finite then $|G| = |L(G)|$.*

Proof. Here we stress that in the associated Lie ring we can only "see" the elements of each γ_i that do not belong to the consecutive term of the lower central series. Hence, it may happen that a non-trivial commutator in G is null in $L(G)$.

If the group satisfies the solubility identity $\delta_d(x_1, \dots, x_{2^d}) = 1$ for every $x_1, \dots, x_{2^d} \in G$, then the same identity holds in $L(G)$. We can only conclude that the derived length of $L(G)$ does not exceed that of G.

However, the fact that the associated Lie ring is generated by $L_1 = \gamma_1/\gamma_2$ gives a precise expression for the terms of the lower central series of $L(G)$, which are

$$\gamma_k(L(G)) = \bigoplus_{i \geq k} \gamma_i/\gamma_{i+1}.$$

This implies that the nilpotency class of $L(G)$ cannot be strictly smaller than that of G. On the other hand, it cannot even be greater than that since every commutator identity $[x_1, \ldots, x_{c+1}] = 1$ in the group is also satisfied by homogeneous elements of $L(G)$ and, by bilinearity, by $L(G)$ itself. □

This construction can be made more general, by taking any strongly central series in place of the lower central series in Definition 2.2. We recall that a strongly central series is a subgroup series

$$G = K_1 \geq K_2 \geq \ldots \geq K_c \geq K_{c+1} = 1,$$

such that $[K_i, K_j] \leq K_{i+j}$ for all $1 \leq i, j \leq c$. However, in this more general setting, the nilpotency class may decrease. Moreover, the subgroups K_i may not be characteristic. For these reasons, since we will use the Lie ring method to bound the nilpotency class of finite groups with regular automorphisms, it will be more convenient for us to work with the classical construction.

For our purpose, we are particularly interested in the automorphisms of $L(G)$ induced by those of G. Let ϕ be an automorphism of G. It naturally acts on every factor group γ_i/γ_{i+1}, since the terms of the lower central series are characteristic subgroups (Theorem 2.1.5). With a little abuse of notation we denote by the same name the Lie ring automorphism induced on $L(G)$. The next theorem gives us more information about it.

Theorem 2.4. *Let G be a group and $L(G)$ its associated Lie ring. Every automorphism ϕ of G induces an automorphism of $L(G)$. If G is finite and ϕ has order coprime to the order of G, then ϕ faithfully acts on $L(G)$ and $|C_{L(G)}(\phi)| = |C_G(\phi)|$.*

Proof. If $\phi \in \mathrm{Aut}(G)$, then for homogeneous elements $l_i + \gamma_{i+1} \in \gamma_i/\gamma_{i+1}$ and $l_j + \gamma_{j+1} \in \gamma_j/\gamma_{j+1}$ we have

$$[l_i+\gamma_{i+1}, l_j+\gamma_{j+1}]^\phi = [l_i, l_j]^\phi+\gamma_{i+j+1} = [l_i^\phi, l_j^\phi]+\gamma_{i+j+1} = [(l_i+\gamma_{i+1})^\phi, (l_j+\gamma_{j+1})^\phi].$$

By definition we extend the action by linearity. This mean that ϕ is an automorphisms of $L(G)$. Its faithfulness is a direct consequence of Corollary 1.18. The claim about the number of its fixed points follows instead from Theorem 1.17. □

We will show an example of the application of this construction in the last section of this chapter, where we study finite groups with a Frobenius group of automorphisms.

2.1.2 Some drawbacks of the associated Lie ring

In the previous section we have seen a standard way to associate a Lie ring to any group. Whilst the construction of the associated Lie ring $L(G)$ from a group G is quite straightforward, still it is not completely without obstacles. This is due to the fact that some properties of G are not properly reflected in $L(G)$. Here we collect the things that may go wrong.

1. Non-isomorphic groups can have isomorphic Lie rings. For instance, consider the dihedral group D_8 and the quaternion group Q_8. They both have a Lie ring with underlying abelian group $(\mathbb{Z}/2\mathbb{Z})^3$. Denoting by $\{u, v, w = [u, v]\}$ a basis of it as an F_2-vector space, then in both cases the structure constants are $[u, v] = w$, $[u, w] = [v, w] = 0$.

2. The derived length may decrease. We construct a nilpotent group G for which $dl(G) = 3$ while $dl(L(G)) = 2$. Consider the group generated by x and y satisfying the relations

   - $x^{11} = y^{11} = 1$,

- $[y, x, y] = 1$,
- $[[y, x, x, x], [y, x, x]] = [[y, x, x, x, x], [y, x]] = 1$,
- $[[y, x, x], [y, x]]^{-1}[y, x, x, x, x, x] = 1$,
- $[[y, x, x, x], [y, x]]^{-1}[y, x, x, x, x, x, x] = 1$.

Let G be its quotient by the 8th term of its lower central series, so that G is nilpotent of class 7 by construction. It can be checked with GAP that G has order 11^8. The derived length of the group equals 3.

The associated Lie ring $L(G)$ has underlying group $(\mathbb{Z}/11\mathbb{Z})^8$. Let $\{e_0, \ldots, e_7\}$ be the basis given by $e_0 = x + \gamma_2$, and $e_i = [y, x^{i-1}] + \gamma_{i+1}$. Then the structure constants are $[e_i, e_0] = e_{i+1}$, for all $1 \leq i \leq 6$ and $[e_i, e_j] = 0$ otherwise. Observe that

$$[e_3, e_1] = [e_4, e_1] = [e_5, e_1] = [e_6, e_1] = 0.$$

This is not immediately evident. It is a consequence of the fact that $[y, x^{i-1}, y] \in \gamma_{i+2}$, for all $i = 3, \ldots, 6$. Nevertheless, those Lie brackets are not involved in the proof that the derived length of $L(G)$ equals 2, which follows easily from the last three items of the list above.

The key point here is to impose relations that "do not respect" the grading, so that for every $x_i \in G^{(1)} \cap \gamma_i$ and $x_j \in G^{(1)} \cap \gamma_j$ for which $[x_i, x_j]$ is non-trivial, we have that $[x_i, x_j] \in \gamma_{i+j+1}$. In particular, to obtain $dl(L(G)) < dl(G)$ we need at least one of such Lie brackets to have $i + j < c$, where c is the nilpotency class of the group.

3. The induced automorphism may be trivial. The group homomorphism $\pi : \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(L(G))$ associates to each automorphism of the group G its induced Lie ring automorphism. In general it has non-trivial kernel, given by those operators that fix all factor groups γ_i/γ_{i+1}.

For instance, every inner automorphism of G induces a trivial automorphism of $L(G)$. An automorphism ϕ coprime to G is never in $\ker \pi$, by Corollary 1.18.

4. The number of fixed points may increase. To every fixed point of ϕ in G, there corresponds a fixed point of the induced automorphism in $L(G)$. In particular, for every term γ_i of the lower central series of G, we have
$$C_{\gamma_i}(\phi)\gamma_{i+1}/\gamma_{i+1} \subseteq C_{\gamma_i/\gamma_{i+1}}(\phi).$$
However, if the action is not coprime, then the inclusion may not be sharp. We have the inequality
$$|C_{\gamma_i/\gamma_{i+1}}(\phi)| \leq |C_{\gamma_i}(\phi)|,$$
by applying Theorem 1.16. This implies that for a nilpotent group G of class c, we can only bound the fixed points of the induced automorphism
$$|C_{L(G)}(\phi)| \leq \prod_{i=1}^{c} |C_{\gamma_i}(\phi)|.$$
Incidentally, this bound shows that if ϕ is regular, then the induced automorphism of $L(G)$ is also regular.

As stated in Theorem 2.4, the last two situations cannot happen if the action is coprime. However, in the general case, the group $\langle\phi\rangle$ acts not necessarily faithfully on $L(G)$. The order of the induced automorphism is then a divisor of the order of ϕ.

2.2 Ground ring extension

In the study of group automorphisms, there is a big advantage in replacing a group with a "linear version" of it, considering for instance its associated

Lie ring. Indeed, as already mentioned, we are allowed to perform operations even when those ones have no correspondence in the language of groups. And since the structure of a Lie ring is more regular than that of a group, we can hope that the problem becomes easier to solve.

Here we see an example of this concept that will be useful in the next chapter. The objective is to extend the ground ring of $L(G)$ so that we have a grading which is linked to the (induced) automorphism.

This technique has been described by Higman who successfully applied it to show the existence of Higman's function (see Theorem 1.8), bounding the nilpotency class of a group or a Lie ring with a regular automorphism of prime order. More recently, it has proved to be extremely effective in the study of finite groups with a Frobenius group of automorphisms (see for instance [MS10, KMS14, KM13a]).

Even if this construction will be eventually applied to the associated Lie ring of a finite group, this holds in great generality, and so it will be presented here. We follow Khukhro's book [Khu93, Chapter 4]. The reader is also referred to [HB82, Chapter 8] for an alternative extensive discussion about it.

Let L be a Lie ring and suppose that ϕ is an automorphism of L of order n. We can consider

$$\tilde{L} = \mathbb{Z}(\omega) \otimes L,$$

where ω is a primitive nth root of unity. This is a $\mathbb{Z}(\omega)$-module and a Lie ring, by setting $k(x \otimes l) = kx \otimes l$ and $[x \otimes l, x' \otimes l'] = xx' \otimes [l, l']$, for all $l, l' \in L$ and $k, x, x' \in \mathbb{Z}(\omega)$. The map ϕ induces an automorphism of $\tilde{L}$ of the same order n, that is

$$(x \otimes l)^{\phi} = x \otimes l^{\phi}.$$

The fact that the order of ϕ equals n implies that its eigenvalues are nth roots of unity, and hence contained in $\mathbb{Z}(\omega)$. Therefore, we can define an analogue of the eigenspace decomposition. For all $i = 0, \dots, n-1$, the submodule

$$\tilde{L}_i = \left\{ l \in \tilde{L} : l^{\phi} = \omega^i l \right\} \tag{2.1}$$

is called the ϕ-homogeneous component of degree i. Observe that $\tilde{L}_0$ corresponds to $C_{\tilde{L}}(\phi)$. The homogeneous components satisfy $[\tilde{L}_i, \tilde{L}_j] \subseteq \tilde{L}_{i+j}$, where the index on the right-hand side is taken modulo n. The next proposition tells us when they form a $(\mathbb{Z}/n\mathbb{Z})$-grading of $\tilde{L}$.

Proposition 2.5 ([Khu98], Lemma 4.1.1)**.** *The following inclusions hold*

$$n\tilde{L} \subseteq \tilde{L}_0 + \tilde{L}_1 + \cdots + \tilde{L}_{n-1} \subseteq \tilde{L}.$$

Moreover, if $l_0 + l_1 + \cdots + l_{n-1} = 0$ for $l_j \in \tilde{L}_j$, then $nl_j = 0$, for all $j = 0, 1 \dots, n-1$. Hence, if the additive group of L has no n-torsion, then $n\tilde{L} = \tilde{L}$ and the sum of the ϕ-homogeneous components is direct.

Proof. For every $l \in \tilde{L}$ and each $i = 0, \dots, n-1$ define $l_i = \sum_{k=0}^{n-1} \omega^{-ik} l^{\phi^k}$. It is easy to check that $l_i \in \tilde{L}_i$. On summing the l_i together we get

$$\sum_{i=0}^{n-1} l_i = \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} \omega^{-ik} l^{\phi^k} = \sum_{k=0}^{n-1} l^{\phi^k} \sum_{i=0}^{n-1} \omega^{-ik} = n l^{\phi^0} = nl.$$

Now assume that any linear combination of elements belonging to different ϕ-homogeneous components is trivial. We can apply to it the automorphisms ϕ^k, for $k = 0, \dots, n-1$. We get an homogeneous system of n equations in $l_0, \dots, l_{n-1}$, which is

$$\begin{aligned}
&l_0 + l_1 + l_2 + \cdots + l_{n-1} = 0\\
&l_0 + \omega l_1 + \omega^2 l_2 + \cdots + \omega^{n-1} l_{n-1} = 0\\
&l_0 + \omega^2 l_1 + \omega^4 l_2 + \cdots + \omega^{2(n-1)} l_{n-1} = 0\\
&\dots\\
&l_0 + \omega^{(n-1)} l_1 + \omega^{2(n-1)} l_2 + \cdots + \omega^{(n-1)(n-1)} l_{n-1} = 0.
\end{aligned}$$

To show that $nl_i = 0$ for some i, we multiply each row by an appropriate power of ω which makes the coefficient of l_i equal to 1. Then we sum all equations. For $j \neq i$ the coefficient of l_j is $\sum_{k=0}^{n-1} \omega^{(j-i)k}$, which equals 0. Hence, we get $nl_i = 0$.

The third statement about the direct sum easily follows. $\square$

We list below the cases where there exists a suitable n for which, given a primitive nth root of unity ω, the extended ring $\tilde{L} = \mathbb{Z}(\omega) \otimes L$ admits a cyclic grading. This is obtained by simply applying the proposition above and the following well-known fact, which we state here without proof.

Lemma 2.6. *Let p be a prime number. If a p-group P acts on a p-group G then $C_G(P) \neq 1$.*

The first case we analyse is that of a p-group.

Lemma 2.7. *Suppose that L is a Lie ring with underlying additive structure of a p-group, for some prime number p. If L admits a fixed-point-free automorphism of finite order n, then the Lie ring $\tilde{L}$, obtained by extending the ground ring by a primitive nth root of unity, has a cyclic grading.*

Proof. This is proved by reduction to the case of a coprime action, from which the statement easily follows by applying Proposition 2.5. Indeed, suppose that $p|n$. Then the Hall p'-subgroup $\langle\psi\rangle$ of the group $\langle\phi\rangle$ acts on L. Moreover, its action is fixed-point-free if that of ϕ is. Indeed if not, by Lemma 2.6, we would have some non-trivial fixed points of the Sylow p-subgroup of $\langle\phi\rangle$ on $C_L(\psi)$, and those would be elements of $C_L(\langle\phi\rangle) = C_L(\phi) = 1$. Therefore, we could consider the automorphism ψ of L of order dividing n and coprime to p. $\square$

Observe that this case includes Lie algebras over finite fields. The next case is that of a Lie ring with a fixed-point-free automorphism of order a prime power.

Lemma 2.8. *Suppose that L is a Lie ring admitting a fixed-point-free automorphism ϕ of order $n = p^k$, for some prime p. Then the Lie ring $\tilde{L}$, obtained by extending the ground ring by a primitive nth root of unity, has a cyclic grading.*

Proof. The p-group of p-torsion points of L must be trivial, since otherwise ϕ would have some non-trivial fixed points on it, by Lemma 2.6. This implies that $p^k L = L$ and the result follows from Proposition 2.5. □

Note that in all cases above a Lie ring with an automorphism of finite order has been replaced by a Lie ring with a finite cyclic grading. The zero-component of the grading is trivial whenever the automorphism is regular. The reverse process is also possible.

Regarding the problem of bounding nilpotency class and derived length of Lie rings with regular automorphisms, it has been particularly fruitful to consider graded Lie rings and apply to them some techniques of additive combinatorics. Kreknin and Kostrikin's results [KK63, Kre63], for instance, are obtained in this way.

In the next section, as another possible application, we will sketch the proof of [KMS14, Theorem 5.8] about finite groups with a metacyclic Frobenius group of automorphisms. There we will also see that it is not always necessary for $\tilde{L}$ to be a graded Lie ring. If some combinatorics can be done in $\sum_{i=0}^{n-1} \tilde{L}_i$, then we can always say something about $n\tilde{L}$ and that might be enough for deducing something about L.

2.3 An application to finite groups with a metacyclic Frobenius group of automorphisms

In this section we see an application of the Lie ring method just discussed. The theorem presented is due to Khukhro, Makarenko and Shumyatsky. We sketch the proof here to show how the associated Lie ring can be useful to the study of finite groups with a Frobenius group of automorphisms. The same ideas will be used and further developed in Chapter 3 to prove a partial generalisation of this result.

Theorem 2.9 ([KMS14], Theorem 5.8). *Let G be a finite group with a Frobenius group of automorphisms FH, with kernel F and complement H, satisfying the following*

1. *$F = \langle \phi \rangle$ is cyclic of order n.*

2. *F acts fixed-point-freely, that is $C_G(F) = C_G(\phi) = 1$.*

3. *$C_G(H)$ is nilpotent of class c.*

Then G is nilpotent of nilpotency class bounded in terms of c and $|H|$.

This result comes from an analogous one about Lie rings.

Theorem 2.10 ([KMS14], Theorem 5.6). *Let FH be a Frobenius group with cyclic kernel F of order n and complement H of order q. Suppose that FH acts by automorphisms on a Lie ring L in such a way that $C_L(F) = 0$ and $C_L(H)$ is nilpotent of class c. Then*

(a) L is soluble of (c, q)-bounded class.

(b) For some functions $u = u(c, q)$ and $v = v(c, q)$, depending on c and q only, the subring $n^u L$ is nilpotent of class v.

(c) Moreover, L is nilpotent of (c, q)-bounded class in any of the following cases:

(i) the additive group of L is periodic,

(ii) L is a Lie algebra,

(iii) n is invertible in the ground ring of L,

(iv) $nL = L$,

(v) n *is a prime-power.*

The Lie ring result is the most elaborate part, while the theorem about finite groups follows from applying this to the associated Lie ring.

Sketch of the proof of Theorem 2.10 . Consider $\tilde{L} = \mathbb{Z}(\omega) \otimes L$, where ω is a primitive nth root of unity. The group FH acts in a natural way on $\tilde{L}$ and its action inherits the two conditions in the statement: $C_{\tilde{L}}(F) = 0$ and $C_{\tilde{L}}(H)$ is nilpotent of class c. Since the conditions $(i) - (v)$ would also hold for $\tilde{L}$, and the conclusion of the theorem for L would follow from the same conclusion for $\tilde{L}$, we can assume that $\tilde{L} = L$ and the ground ring contains ω.

Let ϕ be a generator of F. For each $i = 0, \dots, n-1$, we define the "eigenspace" for ω^i as $L_i = \left\{ l \in L | l^\phi = \omega^i l \right\}$. Then we have

$$nL \subseteq \sum_{i=o}^{n-1} L_i \subseteq L, \qquad \text{and} \qquad [L_i, L_j] \leq L_{i+j},$$

where all indices are taken modulo n. In particular, the zero-component L_0 coincides with $C_L(\phi)$, and consequently it is trivial.

Hereafter we will use the following index convention: the letter l with index $i \in \mathbb{Z}/n\mathbb{Z}$ will denote a homogeneous element of L_i, with the index only indicating which homogeneous component this element belongs to. Different elements can hence be denoted by the same symbol. For instance, l_i and l_i can be different elements of L_i so that $[l_i, l_i]$ is a non-zero element of L_{2i}.

We will mostly work with the FH-invariant subring $\sum_{i=0}^{n-1} L_i$. The sum may not be direct but, by Lemma 2.5, any linear dependence of elements from different L_i is annihilated by n, that is

$$\text{if} \quad l_1 + l_2 + \cdots + l_{n-1} = 0, \qquad \text{then} \qquad nl_1 = \cdots = nl_{n-1} = 0. \quad (2.2)$$

Moreover, the ϕ-homogeneous components are fixed by F and permuted by the (necessarily) cyclic complement H as described below. Let h be a generator of H, of order q. Then, for every $i \in (\mathbb{Z}/n\mathbb{Z})^*$ and $j \in \mathbb{Z}/q\mathbb{Z}$, we have

$$(L_i)^{h^j} = L_{r^j i},$$

where r is a primitive qth root of unity in $\mathbb{Z}/n\mathbb{Z}$.

We use the notation $l_{r^j i} = (l_i)^{h^j}$ with $l_i \in L_{s_i}$ and $X_i = \sum_{j=0}^{q-1} l_{r^j i} \in C_L(h)$. The nilpotency condition about $C_L(H)$ implies that

$$[X_1, \ldots, X_{c+1}] = 0.$$

We can then expand the X_i and consider the resulting linear combination of Lie brackets in $l_{r^j i}$. If $[l_1, \ldots, l_{c+1}]$ is non-zero, then one of the following holds.

1. If the sum of the ϕ-homogeneous components is direct, then there must be another Lie bracket belonging to the same component, that is

$$s_1 + \cdots + s_{c+1} \equiv r^{\alpha_1} s_1 + \cdots + r^{\alpha_{c+1}} s_{c+1} \mod n, \tag{2.3}$$

for some $0 \neq (\alpha_1, \ldots, \alpha_{c+1}) \in (\mathbb{Z}/q\mathbb{Z})^{c+1}$. If $(s_1, \ldots, s_{c+1})$ satisfies the congruence above, then it is said to be *r-dependent*. We call it r-independent otherwise.

2. if $K = \sum_{i=0}^{n-1} L_i$ is not a direct sum, then it might be that there are no other elements in the same ϕ-homogeneous component. However, by Equation (2.2), we have $n[l_1, \ldots, l_{c+1}] = 0$.

Hence, in any case, the subring K satisfies the following combinatorial condition:

$$n[l_1, \ldots, l_{c+1}] = 0, \quad \text{whenever } (s_1, \ldots, s_{c+1}) \text{ is } r\text{-independent}. \tag{2.4}$$

The authors proved that this implies the solubility of L. We refer the interested reader to the original paper for the details.

We now sketch the proof of part (b). We start stating two results that will be used in the argument.

Lemma 2.11. *Let r be a primitive qth root of unity in $\mathbb{Z}/n\mathbb{Z}$. Suppose that for some integer m a sequence $(s_1, \ldots, s_k)$ of non-zero elements of $\mathbb{Z}/n\mathbb{Z}$ contains at least $q^m + m$ different values. Then one can choose an r-independent subsequence $(s_1, s_{i_2}, \ldots, s_{i_m})$ of length m and containing s_1.*

We will use this lemma in order to bound the number of different values appearing in a sequence with no r-independent subsequence of a certain length and starting with a fixed element. We will also need the following result.

Lemma 2.12. *Suppose that $K = \sum_{i=0}^{n-1} L_i$ satisfies condition (2.4). Then there exist (c, q)-bounded numbers l and m such that, for every $b \in \mathbb{Z}/n\mathbb{Z}$, we have*

$$n^l[K, \underbrace{L_b, \ldots, L_b}_{m}] \subseteq [[K, K], [K, K]].$$

We are now ready to prove (b), using induction on the derived length of L. If L is abelian, then there is nothing to prove. Therefore, assume that L is metabelian. In this case we can move around the elements of any Lie bracket starting from the third one. In other words $[x, y, z] = [x, z, y]$, for all $x \in [L, L]$. Now take $g = (m-1)(q^{c+1} + c) + 2$, with m as in Lemma 2.12. For every sequence of g non-zero elements $(s_1, \ldots, s_g)$, consider the Lie bracket $[[K, K]_{s_1}, L_{s_2}, \ldots, L_{s_g}]$, where $[K, K]_{s_1} = [K, K] \cap L_{s_1}$. If the sequence contains an r-independent subsequence of length $c+1$ starting with s_1, then by permuting the L_{s_i}, we can assume that such subsequence is at the beginning of the Lie bracket. Hence $n[[K, K]_{s_1}, L_{s_2}, \ldots, L_{s_g}] = 0$.

If no such subsequence exists, then $(s_1, \ldots, s_g)$ contains at most $q^{c+1} + c$ different values, by Lemma 2.11. The integer g is big enough to guarantee that either the value s_1 occurs at least $m + 1$ times in the sequence, or else, another value different from s_1 occurs at least m times. Again, modulo re-ordering the elements, we can assume $s_2 = \cdots = s_m$. We get that $n^l[[K, K]_{s_1}, L_{s_2}, \ldots, L_{s_g}]$ is trivial, by Lemma 2.12. Thus, we conclude that if

L is metabelian, then $n^l\gamma_{g+1}(K) = 0$. Clearly we can choose a (c, q)-bounded number l_1 such that $\gamma_{g+1}(n^{l_1}K) = 0$. As $nL \subseteq K$, we get $\gamma_{g+1}(n^{l_1+1}L) = 0$.

Now suppose that the derived length of L is at least 3. By induction step there exist (c, q)-bounded numbers l_2 and g' for which $\gamma_{g'+1}(n^{l_2}[L, L]) = 0$. Since $g \geq 4$, we can choose an integer y so that $y(g + 1 - 4) \geq (l_1 + 1)(g + 1)$ and this only depends on c and q. Set $x = \max\{y, \lfloor (l_2 + 1)/2 \rfloor\}$ and $M = n^x L$. We have

$$\begin{aligned}\gamma_{g+1}(M) = \gamma_{g+1}(n^x L) = n^{x(g+1)}\gamma_{g+1}(L) &\subseteq n^{(l_1+1)(g+1)+4x}\gamma_{g+1}(L) \\ &\subseteq n^{4x}[[L, L], [L, L]] = [[M, M], [M, M]],\end{aligned}$$

from the metabelian case. Moreover,

$$\gamma_{g'+1}([M, M]) = \gamma_{g'+1}(n^{2x}[L, L]) \subseteq \gamma_{g'+1}(n^{l_2}[L, L]) = 0.$$

The claim then follows by the Lie ring analogue of Hall's criterion 1.11.

For the complete proof of part (c) we refer the reader to the paper. We stress that conditions $(ii) - (v)$ are exactly the ones seen in Proposition 2.5, Lemmas 2.7 and 2.8. Hence, the only non obvious case is when L is periodic. Since this includes the case of a finite L, which is of interest to us, the argument of case (i) is reproduced below.

We claim that if L is periodic, then we can restrict to the case of a p-group, for some prime p. Indeed, L decomposes into the direct sum of its Sylow p-subgroups T_p. The subgroups T_p are FH-invariant ideals satisfying $[T_p, T_q] = 0$ for $p \neq q$. Hence, the nilpotency of L follows from that of every T_p, and a bound for its nilpotency class is given by the maximum of the bounds for such subgroups. Therefore, the claim follows by applying Lemma 2.7. $\square$

We are now ready to use the result about Lie rings and the construction of the associated Lie ring to prove Theorem 2.9

Sketch of the Proof of Theorem 2.9. First, the nilpotency of G is guaranteed by the nilpotency of $C_G(H)$ and the fixed-point-free action of the Frobenius kernel, as shown in [KMS14, Theorem 2.7]. The proof uses arguments of Representation Theory.

We use the Lie ring method to bound the nilpotency class of G. Its associated Lie ring is

$$L(G) = \bigoplus_{i=1}^{k} \gamma_i/\gamma_{i+1},$$

where k is the nilpotency class of the group and γ_i represent the terms of its lower central series.

The Frobenius group FH naturally induces a (not necessarily faithful) action of the same group on $L(G)$ which inherits conditions 2 and 3. The former follows directly from Theorem 1.16 and the latter is a straightforward consequence of Theorem 1.21. Indeed, if the fixed points of H in $L(G)$ are covered by the fixed points of H in G, then every commutator identity holding in $C_G(H)$ also holds in $C_{L(G)}(H)$. Specifically, this arguments only proves that the nilpotency class of $C_{L(G)}(H)$ does not exceed that of $C_G(H)$, but this will be enough for our purpose. Nevertheless, since we do not require coprimality between $|G|$ and $|F|$, it may happen that the order of ϕ over $L(G)$ reduces to a proper divisor of n. In any case, construct the s uitable extended Lie ring $L = \mathbb{Z}(\omega) \otimes L(G)$, where ω is a primitive mth root of unity and m the order of ϕ on $L(G)$. The natural action of FH on L inherits once again conditions 2 and 3. We can now simply apply Theorem 2.10. The conclusion follows from (i) of part (c). $\square$

It worth mentioning few points that appear in this proof and are almost standard in this kind of problems. First, the Lie ring method cannot help in proving the nilpotency of the finite group. If not part of the hypotheses, this must be deduced by arguments of different nature. The Lie ring technique, which relies on it, is only applied to produce a bound for the nilpotency class or the derived length.

The result for groups comes from an analogous one for Lie rings. The latter is usually the most elaborate part.

Because of the Hall's criterion, the key case is when the Lie ring is metabelian. Indeed, if in this case we can bound the nilpotency class by a function of certain variables, then a function of the same variables bounds the nilpotency class in case of any derived length.

Chapter 3

Frobenius groups of automorphisms with abelian kernel

In this chapter we look at the case of a Frobenius group of automorphisms FH having a fixed-point-free abelian non-cyclic kernel F. Khukhro, Makarenko and Shumyatsky proved that a finite group G admitting such a group of automorphisms is nilpotent, whenever the centraliser $C_G(H)$ of the Frobenius complement is nilpotent as well. This is, in fact, a special case of [KMS14, Theorem 2.7] that guarantees the nilpotency of the group whenever the Frobenius kernel acts fixed-point-freely on it and the centraliser of the complement is nilpotent, hence dropping any restriction on the nilpotency class of the kernel.

The same authors showed that the nilpotency class of G is generally unbounded even for a fixed group FH and cyclic $C_G(H)$, so that the hypothesis about the cyclicity of the Frobenius kernel seen in the previous chapter is essential. For instance, consider a Lie ring L whose additive group is the direct sum of three copies of $\mathbb{Z}/p^m\mathbb{Z}$, for some prime $p \neq 2$. Let e_1, e_2, e_3 be its generators whose structure constants are given by $[e_1, e_2] = pe_3$, $[e_2, e_3] = pe_1$ and $[e_3, e_1] = pe_2$. Then L admits a Frobenius group of automorphisms FH with non-cyclic kernel F of order 4 and complement H of order 3 such that $C_L(F) = 0$ and $C_L(H)$ is abelian (since one-dimensional). Specifically, the action of $F = \{1, f_1, f_2, f_3\}$ is $f_i(e_i) = e_i$ and $f_i(e_j) = -e_j$, for $j \neq i$. The complement $H = \langle h \rangle$ acts in such a way that $h(e_i) = e_{i+1}$, where the index has to be taken modulo 3 when it is greater than 3. It is easy to see that L has nilpotency class exactly m.

If we choose $p > m$, then the Lazard correspondence based on the Baker-Campbell-Hausdorff formula transforms L into a finite p-group P with the same nilpotency class and same group of automorphisms FH such that $C_P(F) = 1$ and $C_P(H)$ is cyclic.

Nevertheless, there are some cases where the Frobenius kernel is abelian non-cyclic but a bound for the nilpotency class is still possible. For instance, in [KS11] it is proved that if a Frobenius group FH, with abelian kernel F of rank at least three, acts coprimely on a finite group G in such a way that $C_G(H)$ is abelian and for every f in $F \setminus \{1\}$ the centraliser $C_G(f)$ is nilpotent of class at most c, then G has nilpotency class bounded in terms of c and $|H|$.

Here we prove a similar result in case H is relatively small compared to F.

Theorem 3.1. *Let G be a finite group admitting a Frobenius group of automorphisms FH, with abelian kernel F and complement H of order q. Suppose that $C_G(H)$ is abelian and F acts coprimely on G in such a manner that $C_G(F)$ is trivial. If q^2 is smaller than the least prime divisor of $|F|$, then G is nilpotent of q-bounded class.*

The proof is based on a Lie ring method. The first section sets some preliminary facts. The second one analyses graded Lie rings with few non-trivial components and many commuting components. There we prove an instrumental proposition. The last section is devoted to the proof of the theorem.

3.1 Preliminaries

We recall the notation and some general facts. In what follows we use the term "span" both for the subspace (in the case of algebras) and for the additive subgroup generated by a given subset of a Lie ring. For subsets X and Y contained in a Lie ring L and $t \geq 2$, we write $[X, {}_t Y]$ for $[[X, {}_{t-1}Y], Y]$. We denote by $\langle U \rangle$ the Lie subring generated by a subset U, and by ${}_{id}\langle U \rangle$ the ideal generated by U.

For graded Lie rings we use the usual index convention: by the symbol l_a we will denote a homogeneous element in the graded component L_a with the index only indicating which component this element belongs to. So different elements can be denoted by the same symbol.

We use the abbreviation, say, "(m, n)-bounded" for "bounded from above by a function depending on m and n only".

We shall need the following combinatorial lemma, generalising [Khu93, Lemma 4.2.5].

Lemma 3.2. *Let A be a finite abelian group, written additively. Let $a_1, \cdots, a_k$ be not necessarily distinct elements of $A \setminus \{0\}$. Consider the set*

$$M = \left\{ \sum_{s \in S} a_s \mid S \subseteq \{1, \cdots, k\} \right\},$$

where, by definition, the sum is 0 for $S = \emptyset$. Then $|M| \geq \min(k+1, p)$, where p is the least prime divisor of $|A|$.

Proof. We proceed by induction on k. The case $k = 1$ is obvious, since $a_1 \neq 0$. By inductive hypothesis the set

$$M' = \left\{ \sum_{s \in S} a_s \mid S \subseteq \{1, \cdots, k-1\} \right\}$$

has cardinality $|M'| \geq \min(k, p)$. Since $M = M' \cup (M' + a_k)$, we have $|M| \geq |M'|$. If the inequality is strict, then

$$|M| \geq min(k, p) + 1 \geq min(k + 1, p).$$

If not, then $M' + a_k = M'$, so that M' is invariant under adding elements of $\langle a_k \rangle$. Therefore, the cardinality of M' is not smaller than p, and the statement follows. $\square$

We will use the following generalisation of Proposition 2.5, regarding the action of abelian groups on Lie rings. We recall that a finite abelian group is isomorphic to its group of characters. For the proof of this fact and other standard results about characters of abelian groups, we refer the reader to [Hup98].

Proposition 3.3. *Let L be a Lie ring. Let G be an abelian group of order n with character group $\hat{G}$. Suppose that G acts on L and define $\tilde{L} = \mathbb{Z}(\omega) \otimes L$ to be the ring obtained from L by extending the ground ring by a primitive nth root of unity ω. Then the following inclusion holds*

$$n\tilde{L} \subseteq \sum_{\chi \in \hat{G}} L_\chi \subseteq \tilde{L},$$

where $L_\chi = \left\{ l \in \tilde{L} \mid l^g = \chi(g)l \text{ for all } g \in G \right\}$. Moreover, if $l_1 + \cdots + l_n = 0$ for $l_j \in L_{\chi_j}$, then $nl_j = 0$ for all $j = 1, \ldots, n$. Hence if the additive group of $\tilde{L}$ has no n-torsion, then $\tilde{L}$ is $\hat{G}$-graded.

Proof. Since G and $\hat{G}$ are isomorphic, they have in particular the same order. For $i = 1 \ldots, n$ we call χ_i the elements of $\hat{G}$. For convenience in this proof we use the multiplicative notation for G.

Consider $l \in \tilde{L}$ and define $l_i = \sum_{g \in G} \chi_i(g^{-1}) l^g$. It is not hard to see that $l_i \in L_{\chi_i}$. Indeed for every $h \in G$ we have

$$l_i^h = \left(\sum_{g \in G} \chi_i(g^{-1}) l^g \right)^h = \sum_{g \in G} \chi_i(g^{-1}) l^{gh} = \sum_{g \in G} \chi_i(hg^{-1}) l^g = \chi_i(h) l_i.$$

We now prove the inclusion $n\tilde{L} \subseteq \sum_{i=1}^{n} L_{\chi_i}$. Let l_i be defined as above. Summing over i we get

$$\sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \sum_{g \in G} \chi_i(g^{-1}) l^g = \sum_{g \in G} l^g \sum_{i=1}^{n} \chi_i(g^{-1}) = nl,$$

where the last equality follows from the fact that $\sum_{\chi \in \hat{G}} \chi(g) \neq 0$ only when $g = 1$ and in this case the sum equals the order of G.

Suppose that a linear combination of elements belonging to different homogeneous components is trivial, that is $l_1 + l_2 + \cdots + l_n = 0$, where $l_i \in L_{\chi_i}$. We can apply $g_1, \ldots, g_n$ to it, where g_i are the elements of G. Then we get the following equations

$$\begin{aligned}
\chi_1(g_1)l_1 + \chi_2(g_1)l_2 + \cdots + \chi_n(g_1)l_n &= 0 \\
\chi_1(g_2)l_1 + \chi_2(g_2)l_2 + \cdots + \chi_n(g_2)l_n &= 0 \\
&\cdots \\
\chi_1(g_n)l_1 + \chi_2(g_n)l_2 + \cdots + \chi_n(g_n)l_n &= 0.
\end{aligned}$$

In order to prove that $nl_i = 0$, we multiply the jth equation by $\chi_i(g_j^{-1})$ and then we sum them up. The coefficient of l_k is $\sum_{j=1}^{n} \chi_k(g_j)\chi_i(g_j^{-1})$, which is equal to n if $i = k$ and 0 otherwise. Hence, if L has no n-torsion, then the sum of the homogeneous components is direct and the last statement easily follows. $\square$

We will also require a Lie ring analogue of Hall's criterion for nilpotency (Theorem 1.11) which we state here without proof.

Theorem 3.4. *If a Lie ring L has an ideal K such that K is nilpotent of class k and $L/[K,K]$ is nilpotent of class c, then L is nilpotent and its nilpotency class is bounded by $f(k,c) = (c-1)\frac{k(k+1)}{2} + k$.*

The next result is due to N. Makarenko. It is about Lie rings graded over a cyclic group with a given number of non-zero component.

Proposition 3.5 (Proposition 1 [Mak07]). *Let L be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring with d non-trivial components. There exists a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that the $f(d,t)$th term of the derived series $L^{(f(d,t))}$ is contained in the subring generated by the set $[L,\, {}_tL_0]$.*

We shall use this proposition in the next section, in the situation where $[L,\, {}_tL_0] = 0$, so that L is soluble of $f(d,t)$-bounded derived length. The next lemma will be used to prove the existence of such positive integer t.

Lemma 3.6. *Let L be a Lie ring and K a nilpotent subring of class c generated by the subsets $K_1, \dots, K_d$. Suppose that there exists a number n such that $[L,\, {}_nK_i] = 0$ for each $i = 1, \dots, d$. Then there exists a (c,d,n)-bounded number g such that $[L,\, {}_gK] = 0$.*

Proof. First observe that $[L,\, {}_gK]$ is generated by simple Lie brackets of the form $[l, k_{i_1}, \dots, k_{i_r}]$, where $l \in L$, $k_{i_j} \in K_{i_j}$ for $i_j \in \{1, \dots, d\}$, and $r \geq g$. Hence, the statement is proved if we show that all these Lie brackets vanish. This is done by induction on c. Suppose that $c = 1$, so that $K' = 0$. Take $g = d(n-1)+1$. Since for any permutation π of $\{1, \cdots, r\}$ we have

$$[L, K_{i_1}, \dots, K_{i_r}] \leq [L, K_{i_{\pi(1)}}, \dots, K_{i_{\pi(r)}}] + [L, K'],$$

we can rearrange the elements of K without changing the Lie bracket. The integer g is big enough to ensure that there are at least n elements among the k_{i_j} belonging to the same K_i. Consequently, the Lie bracket can be written as $[l, k_{i_{\pi(1)}}, \dots, k_{i_{\pi(n)}}, *, \dots, *]$, where $k_{i_{\pi(1)}}, \dots, k_{i_{\pi(n)}} \in K_i$ and the asterisks denote the remaining elements in the K_{i_j} which are of no consequence, in view of the fact that $[L,\, {}_nK_i] = 0$.

Suppose now that $c > 1$. Since the subring K' has nilpotency class at most $c-1$, the inductive hypothesis ensures that there exists a (c,d,n)-bounded number g_1 for which $[L,\, {}_{g_1}K'] = 0$. We want to prove that

$$[L, \underbrace{K, \dots, K}_{u(d(n-1)+1)}] \leq [L, \underbrace{K', \dots, K'}_{u}],$$

for every positive integer u. We proceed by induction on u, the base step for $u = 1$ being proved already. Hence suppose that $u > 1$. By inductive hypothesis, we can write

$$\begin{aligned}[L, \underbrace{K, \ldots, K}_{u(d(n-1)+1)}] &\leq [L, \underbrace{K', \ldots, K'}_{u-1}, \underbrace{K, \ldots, K}_{d(n-1)+1}] \\ &\leq [L, \underbrace{K', \ldots, K'}_{u}] + [L, \underbrace{K', \ldots, K'}_{u-1}, \underbrace{K_i, \ldots, K_i}_{n}, *, \ldots, *],\end{aligned}$$

and this proves the desired equation. The statement then follows by taking $g = g_1(d(n-1)+1)$. □

The bound for the integer g in the above lemma can be made explicit. It is not hard to see that it grows together with c, d and n. This result will be used in the situation where we only know an upper bound for the number d of subgroups generating the nilpotent subring. Then the monotonicity in this variable will ensure the validity of the bound for g thus obtained.

3.2 Lie rings with many commuting components

This section is devoted to the proof of the following result about graded Lie rings with many commuting components.

Proposition 3.7. *Let A be an additively written abelian group of finite order n. Let $L = \bigoplus_{a \in A} L_a$ be an A-graded Lie ring which satisfies the following properties:*

1. $L_0 = 0$;
2. *L satisfies the m-*condition*: there exists a natural number m such that, if $N_a = \{b \in A \mid [L_b, L_a] \neq 0\}$, then $|N_a| \leq m$ for all a in A;*
3. *$m < p$, where p is the least prime divisor of the order of A.*

Then the following hold:

(a) *L is soluble of m-bounded derived length.*

(b) *L is nilpotent of m-bounded nilpotency class.*

This proposition generalises [Khu08, Theorem 1] and is instrumental in the proof of Theorem 3.1. First we present some preliminary lemmas that hold for Lie rings satisfying the m-condition. These can be already found in Khukhro's article, where they were applied to the case of a cyclic grading.

Lemma 3.8. *Suppose that L is a graded Lie ring satisfying the m-condition. Then the homogeneous subring $\langle L_i \mid i \in S\rangle$ generated by $|S| = s$ homogeneous components has at most $s(m+1)$ non-trivial homogeneous components.*

Proof. The additive group of L is generated by homogeneous simple Lie brackets with elements $x_i \in L_i$, for $i \in S$. Any such Lie bracket of weight greater than 1 can be written as $[y_j, x_i]$, where $y_j \in L_j$ is the initial segment. For this Lie bracket to be non-zero we need that $j \in N_i$. Hence, for a given i, we have at most m possibilities for j and this gives at most ms possibilities for the sum $i + j$. Together with the single homogeneous elements (which represent the Lie brackets of weight 1) we have at most $s + ms$ possibilities for the index of a non-zero homogeneous element of the subring. □

This lemma will be used together with the next one.

Lemma 3.9. *Suppose that L is a graded Lie ring satisfying the m-condition. The homogeneous ideal ${}_{id}\langle L_i\rangle$ generated by any homogeneous component L_i is contained in the subring $\langle L_i, L_k \mid k \in N_i\rangle$.*

Proof. The additive subgroup of the ideal is generated by the simple Lie brackets $[x_i, l_{i_1}, \dots, l_{i_t}]$, where $x_i \in L_i$ and $l_{i_s} \in L_{i_s}$. We use double induction to prove that these Lie brackets belong to the desired subring.

The first induction parameter is t, the weight. For a given t, we proceed by induction on the position of the first (from the left) index not belonging to N_i. If no such index exists, then the Lie bracket obviously belongs to the subring. The assertion is trivially true for $t = 0$. Hence suppose that $t = 1$. If $i_1 \in N_i$, then the desired conclusion holds. If instead $i_1 \notin N_i$, then the Lie bracket $[x_i, l_{i_1}]$ equals zero.

Now assume that $t > 1$ and let i_j be the first index not belonging to N_i. If $j = 1$, then the whole Lie bracket is trivial. Otherwise, we can transpose the entry and write

$$[x_i, \ldots, l_{i_{j-1}}, l_{i_j}, \ldots, l_{i_t}] = [x_i, \ldots, [l_{i_{j-1}}, l_{i_j}], \ldots, l_{i_t}] + [x_i, \ldots, l_{i_j}, l_{i_{j-1}}, \ldots, l_{i_t}].$$

The first term on the right-hand side is in the subring $\langle L_i, L_k \,|\, k \in N_i \rangle$ by the first inductive hypothesis, on the weight of the Lie bracket. Indeed, it is shorter since $[l_{i_{j-1}}, l_{i_j}]$ is regarded as a single entry. The second term is in the subring by the second inductive hypothesis. $\square$

As a consequence, using Lemma 3.8, we get that the ideal generated by any homogeneous component has at most $(m+1)^2$ homogeneous components. We are now ready to prove the proposition.

Proof of Proposition 3.7. This is done by induction on the rank r of A. For a given r, we will first prove the solubility, that is part (a). We will then show the nilpotency (part (b)) by using (a) and the nilpotency for $r - 1$.

If $r = 1$, then A is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Hence, L is soluble of m-bounded derived length by Khukhro's result [Theorem 1, [Khu08]]. Specifically, let $k \in A$ be an arbitrary grading index, and consider the homogeneous ideal ${}_{id}\langle L_k \rangle$. By the previous lemmas, it has at most $(m + 1)^2$ non-trivial components, and therefore, there are at most $m(m + 1)^2$ components that do not centralise it. Let Z_k be the set of all indices $j \in A$ such that $[L_j, {}_{id}\langle L_k \rangle] = 0$ and define the homogeneous ideal $I_k = {}_{id}\langle L_j \,|\, j \in Z_k \rangle$ which centralises ${}_{id}\langle L_k \rangle$.

Then the quotient ring L/I_k has at most $m(m+1)^2$ non-trivial components in the induced grading and its zero-component is trivial.

By Shalev's theorem (Theorem 1.13), it is soluble of m-bounded derived length, namely there exists a function $f(m)$ for which $L^{(f(m))} \subseteq I_k$. We have that $[L^{(f(m))}, L_k] = 0$ and, since this holds for every k, we conclude that $[L^{(f(m))}, L] = 0$ and L is soluble of derived length at most $f(m)+1$.

We now proceed by induction on the derived length of L to get a bound for its nilpotency class. Suppose that L is metabelian. We claim that L is nilpotent of class at most $m+2$. To prove this, it is sufficient to show that every homogeneous simple Lie bracket $\left[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_{m+3}}\right]$ of weight $m+3$ vanishes. If $i_1 + i_2 = 0$ in A, then $[x_{i_1}, x_{i_2}] = 0$, by hypothesis 1. Hence, assume that $i_1 + i_2 = y$, where $y \in A \setminus \{0\}$. We can write the Lie bracket as $\left[x_y, x_{i_3}, \dots, x_{i_{m+3}}\right]$, where $x_y = [x_{i_1}, x_{i_2}] \in L' \cap L_y$. Since L is metabelian, we can arbitrarily rearrange the entries $x_{i_3}, \dots, x_{i_{m+2}}$ and the set of their possible sums M, as it is defined in Lemma 3.2, has cardinality not smaller than $m+1$, by hypothesis 3. Note that the same is true for $y+M$.

Hence, by hypothesis 2, there exists at least an element in $y+M$ which does not belong to $N_{i_{m+3}}$. Let it be given by $y+i_{\sigma(1)}+\cdots+i_{\sigma(s)}$, for a suitable permutation σ of $\{3, \cdots, m+2\}$ and $s \leq m+2$. Then the Lie bracket

$$\left[x_y, x_{i_3}, \dots, x_{i_{m+3}}\right] = \left[x_y, x_{i_{\sigma(3)}}, \dots, x_{i_{\sigma(s)}}, x_{i_{m+3}}, x_{i_{\sigma(s+1)}}, \dots, x_{i_{\sigma(m+2)}}\right]$$

is clearly trivial.

Suppose now that the derived length of L is $d > 2$. Since all hypotheses hold for L' and $L/L^{(2)}$ as well, then by inductive hypothesis we can assume that L' is nilpotent of m-bounded class. The quotient ring $L/L^{(2)}$ is metabelian, and therefore nilpotent of class not greater than $m+2$, by the base case shown above. By Hall's Criterion for Lie rings (Theorem 3.4), we conclude that L is nilpotent of m-bounded class.

Now suppose that A has rank $r > 1$ and assume, by inductive hypothesis, that the two claims have been proved in case of abelian groups of rank up to $r-1$. We create a new grading on L. Choose a subgroup $B < A$ of rank $r-1$ and such that $\bar{A} = A/B$ is cyclic. We can define an $\bar{A}$-grading of L by setting for each $\bar{a} \in \bar{A}$ the corresponding homogeneous component

$$L_{\bar{a}} = \bigoplus_{a \in \bar{a}} L_a.$$

In particular, the zero-component $L_{\bar{0}} = L_B = \oplus_{b \in B} L_b$.

Note that each component of the new grading is, in fact, a direct sum of some original homogeneous components. It follows that subrings and ideals of L that were homogeneous with respect to the original grading will remain so with respect to the new one. Consequently, quotient rings with an (induced) A-grading admit an $\bar{A}$-grading as well.

Let I_k be the ideal defined in the first paragraph. By the remark above, I_k and $\bar{L} = L/I_k$ admit an $\bar{A}$-grading. The number of homogeneous components of $\bar{L}$ in the new grading is m-bounded. Indeed, the number of non-trivial components in the new grading may only become smaller than in the original one. Its zero-component $\bar{L}_{\bar{0}} = \oplus_{b \in B} \bar{L}_b$ is B-graded and satisfies the three hypotheses, the last one due to Lagrange's Theorem. We can then apply the inductive hypothesis to conclude that $\bar{L}_{\bar{0}}$ is nilpotent of m-bounded class.

The idea is now to apply Lemma 3.6. In order to do so, we need to show that $\bar{L}_{\bar{0}}$ is generated in the old grading by m-bounded homogeneous components $\bar{L}_j$ (for $j \in A$), for which there exists an m-bounded number d such that

$$[\bar{L},\, {}_d\bar{L}_j] = [\bar{L}, \underbrace{\bar{L}_j, \ldots, \bar{L}_j}_{d}] = 0. \tag{3.1}$$

Certainly, $\bar{L}_{\bar{0}}$ is generated by $\bar{L}_b$, for $b \in B$. On the other hand, the number of non-trivial components generating $\bar{L}$ is m-bounded, hence the same must hold for its subring. This, of course, is significant when the order of B is large enough (precisely $|B| > m(m+1^2)$).

Identity (3.1) easily follows if we prove that $[L,\, {}_dL_j] = 0$, for all $j \in B$ and some m-bounded number d. This is a consequence of Lagrange's Theorem and hypothesis 3. Indeed, for every $a \in A$ and $b \in B$, the elements $a, a+b, \ldots, a+mb$ are distinct and therefore, at least one among them is not contained in N_b. It follows that $[L_a, {}_{(m+1)}L_b] = 0$. By Lemma 3.6, we conclude that there exists an m-bounded number g for which $[\bar{L}, {}_g\bar{L}_{\bar{0}}] = 0$.

We can now use Proposition 3.5 to deduce that $\bar{L}$ is soluble of m-bounded derived length. Therefore, there exists a function $f(m)$ for which $L^{(f(m))} \subseteq I_k$. This implies that $[L^{(f(m))}, L_k] = 0$ and, since this hold for every k, we conclude that $[L^{(f(m))}, L] = 0$ and L has derived length not greater than $f(m)+1$.

We can now proceed by induction on the derived length of L to get a bound for its nilpotency class. We do not write the details here, as the proof of this fact would essentially replicate the argument seen above when the rank equals 1. $\square$

An alternative and shorter proof of Proposition 3.7 could be obtained by generalising Shalev's Theorem 1.13 to abelian groups (in place of the multiplicative group of complex numbers). However, it is not obvious how to do that as this would involve a completely new proof. Indeed, the original argument relies on the concept of good order. Following Shalev's definition, given a group G and a set $S \subseteq G$, a linear order $<$ on S is called good if there are no $x, y \in S$ such that $xy \in S$ and $x < xy < y$. Most groups do not admit a good order and a typical example is already given by the elementary abelian p-group $(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z})$.

3.3 Groups and Lie rings with a Frobenius group of automorphisms

This section is devoted to the proof of Theorem 3.1.

Proof of Theorem 3.1. Let FH be a Frobenius group with abelian kernel F of order n and complement H of order q. Suppose that FH acts on a finite group G, whose order is coprime to n, in such a manner that $C_G(F) = 1$ and $C_G(H)$ is abelian. We recall that G is nilpotent by [KMS14, Theorem 2.7]. Hence, consider its associated Lie ring

$$L = \bigoplus_{i=1}^{c} \gamma_i/\gamma_{i+1},$$

where c is the nilpotency class of G and the γ_i represent the terms of its lower central series. The action of the group FH on G induces an action of the same group on L. By Lemma 1.20, the kernel F acts fixed-point-freely on L. Moreover, by Theorem 1.21, we have

$$C_L(H) = \bigoplus_{i=1}^{c} C_{\gamma_i}(H)\gamma_{i+1}/\gamma_{i+1}.$$

This implies that $C_L(H)$ is abelian. Since the nilpotency class of L coincides with that of G, the claim will follow if we prove that L has q-bounded nilpotency class.

Consider the extended ring $\tilde{L} = \mathbb{Z}(\omega) \otimes L$, where ω is a primitive nth root of unity. The group FH acts naturally on $\tilde{L}$ and its action inherits the conditions $C_{\tilde{L}}(F) = 0$ and $C_{\tilde{L}}(H)$ is abelian. Since the conclusion of the theorem for L would follow from the same conclusion for $\tilde{L}$, we can assume that $\tilde{L} = L$ and the ground ring contains ω.

Let $\hat{F}$ be the character group of F. Since F and $\hat{F}$ are isomorphic groups, we use the additive notation for the latter as well. Since the action of F is coprime, by Proposition 3.3 L is $\hat{F}$-graded

$$L = \bigoplus_{\chi \in \hat{F}} L_\chi,$$

where $L_\chi = \left\{ l \in L \mid l^f = \chi(f)l, \ \forall f \in F \right\}$. In particular $L_0 = C_L(F) = 0$.

We define an action of H on $\hat{F}$. We set $\chi^h(f) = \chi(f^h)$. It is not hard to prove that the semidirect product $\hat{F}H$ is a Frobenius group. Indeed the fixed-point-free action of every non-trivial element h on $\hat{F}$ follows from that of the same element on F. The group H permutes the homogeneous components of L according to the formula $(L_\chi)^h = L_{\chi^{h^{-1}}}$. Indeed, for l in L_χ and h in H we have

$$(l^h)^f = l^{hfh^{-1}h} = (l^{f^{h^{-1}}})^h = \chi(f^{h^{-1}})l^h.$$

We shall prove that the hypotheses of Proposition 3.7 are satisfied, so that this result is a straightforward consequence of the one for graded Lie rings with many commuting components. We have already mentioned that the zero-component L_0 corresponds to the centraliser of F in L and it is trivial. We now prove that the abelianity of $C_L(H)$ ensures that L satisfies the m-condition, where m is a function of q. For every homogeneous element $l_\alpha \in L_\alpha$ the element $\sum_{h\in H}(l_\alpha)^h$ is in $C_L(H)$. Hence, taking l_α in L_α and l_β in L_β, we can write

$$\left[\sum_{h\in H}(l_\alpha)^h, \sum_{k\in H}(l_\beta)^k\right] = 0.$$

If $[l_\alpha, l_\beta]$ is a non-trivial element of $L_{\alpha+\beta}$, then there must be another element in the same homogeneous component, that is

$$\alpha + \beta = \alpha^{h^{-1}} + \beta^{k^{-1}}, \tag{3.2}$$

for some $h, k \in H$ not all equal to 1. Then necessarily $k \neq 1$, since otherwise $\alpha = \alpha^{h^{-1}}$ which is impossible for $\alpha \neq 0$ and $h \neq 1$.

For each $h \in H$, the map $\phi_h : \chi \to \chi^h - \chi$ is an automorphism of $\hat{F}$. Indeed, it is injective, and so bijective, since h has no non-trivial fixed point in $\hat{F}$. We can write condition 3.2 in this way

$$\beta = \phi_{k^{-1}}^{-1}(\phi_{h^{-1}}(-\alpha)).$$

Since there are q possibilities for h and $q-1$ for k, then it follows that for every $\alpha \neq 0$ there are at most $q(q-1)$ possibilities for β to satisfy this condition and, hence, the component L_α commutes with all but at most $q(q-1)$ components. The last hypothesis of Proposition 3.7 is then satisfied as well. The application of this proposition concludes the proof. $\square$

This proof concludes the chapter. Note that the result for finite groups comes directly from the result for graded Lie rings by using the associated Lie ring.

Chapter 4

On the nilpotency class of some finite groups with a Frobenius group of automorphisms

As an application of Lie ring methods, back in Section 2.3, we have sketched the proof of the following result.

Theorem 4.1 (Khukhro, Makarenko, Shumyatsky). *Let G be a finite group admitting a Frobenius group of automorphisms FH, with cyclic kernel F and complement H. If the kernel acts fixed-point-freely and $C_G(H)$ is nilpotent of class c, then G is nilpotent and its nilpotency class is bounded in terms of c and $|H|$ only.*

Until now, the question whether the bound for the nilpotency class of G could be made independent of the order of the Frobenius complement H remained unsolved. Indeed, there were no explicit examples showing the opposite. In this chapter, we shall give a negative answer to this question by proving the following

Theorem 4.2. *There exists a family $\mathfrak{G}$ of finite nilpotent groups, of unbounded nilpotency class, whose members G satisfy the conditions:*

1. G admits a metacyclic Frobenius group of automorphisms;

2. the centraliser of the Frobenius kernel in G is trivial;

3. the centraliser of the Frobenius complement in G is abelian.

This result is based on an analogous one for Lie algebras. In this case, the transition to finite groups is obtained from the Lazard correspondence.

The main part of this chapter is devoted to proving the result for Lie algebras. This will consist of the explicit construction of a family $\mathfrak{L}$ of Lie algebras, satisfying analogous properties. As a preliminary step, for every prime number p we will construct a Lie algebra L_p which admits a metacyclic Frobenius group of automorphisms $C_p \rtimes C_{p-1}$. Its definition and properties will be discussed in Section 4.1. In particular, we shall see that every Lie algebra L_p is $\mathbb{Z}$-graded. In the same section, we will also calculate the exact dimension of its homogeneous components (Proposition 4.4).

We shall consider I_p and J_p, which are the smallest $C_p \rtimes C_{p-1}$-invariant ideals generated respectively by $C_{L_p}(C_p)$ and by the derived subalgebra of $C_{L_p}(C_{p-1})$. We will study the former in Section 4.2 and the latter in Section 4.3. Since they are both homogeneous, we will study their intersections with the homogeneous components, bounding the dimension of the subspaces thus obtained (Propositions 4.7 and 4.9).

In Section 4.4 we will build the family $\mathfrak{L}$, whose elements are the quotient Lie algebras $L_p/(I_p+J_p)$. We will prove the existence, for any natural number n, of a prime p such that the corresponding Lie algebra $L_p/(I_p+J_p)$ in $\mathfrak{L}$ has nilpotency class at least n. For this part, the previous bounds in Propositions 4.4, 4.7 and 4.9 will be required. We will finally discuss the conditions under which the Lazard correspondence can be applied.

4.1 The family of Lie algebras L_p

A crucial step towards the proof of Theorem 4.2 consists of the following

Proposition 4.3. *There exists a family $\mathfrak{L}$ of nilpotent Lie algebras, of unbounded nilpotency class, such that any member L satisfies the conditions:*

1. *L admits a metacyclic Frobenius group of automorphisms;*
2. *the centraliser of the Frobenius kernel in L is trivial;*
3. *the centraliser of the Frobenius complement in L is abelian.*

This proposition will be proved in Section 4.4. In the same section we will also see that, in order to produce the analogous result for groups, we will need to consider the case where the characteristic of the underlying field changes with L.

This section instead is devoted to the definition and the study of another family of Lie algebras, which is instrumental in the construction of $\mathfrak{L}$. For every prime number p, we will define a $\mathbb{Z}$-graded Lie algebra L_p, as a quotient of a free metabelian Lie algebra. In Proposition 4.4, we will determine the exact dimension of each homogeneous component. Furthermore, we will construct a metacyclic Frobenius group acting by automorphisms on L_p. We use the usual notations $\langle U\rangle$ and ${}_{id}\langle U\rangle$ respectively for the Lie subalgebra and the ideal generated by a subset U.

Let K be a field of characteristic coprime with p, containing a primitive pth root of unity. For generality, we will not specify its characteristic at this stage, because the whole construction works independently of it. However, we stress that, although this is not made explicit by notation, the characteristic may change together with the prime p.

Let $X_p = \{x_1, \ldots, x_{2p-2}\}$ be an ordered set and let $M(X_p)$ denote the free metabelian Lie algebra over K with free generating set X_p. Following

[Bak87], we know that $M(X_p)$ has a basis consisting of X_p together with the left-normed Lie brackets of the form

$$[x_{i_1}, x_{i_2}, \ldots, x_{i_k}], \text{ with } k \geq 2, \text{ and } x_{i_1} > x_{i_2} \leq \ldots \leq x_{i_k}, \tag{4.1}$$

where $x_{i_1}, \ldots, x_{i_k} \in X_p$. In particular, for every fixed number k, the above set represents a basis for the homogeneous component of $M(X_p)$ of weight k, which is the K-vector space spanned by all Lie brackets of length k involving the generators.

Renaming the generators so that $X_p = \{a_1, \ldots, a_{p-1}, v_1, \ldots v_{p-1}\}$ and $a_1 < \ldots < a_{p-1} < v_1 < \ldots < v_{p-1}$, we get that the set in (4.1) consists of the following Lie brackets:

- $\left[a_{i_1}, a_{i_2}, \ldots, a_{i_j}, v_{i_{j+1}}, \ldots, v_{i_k}\right]$, for all $2 \leq j \leq k$, and $i_1 > i_2 \leq \ldots \leq i_j$, $i_{j+1} \leq \ldots \leq i_k$,
- $\left[v_{i_1}, a_{i_2}, \ldots, a_{i_j}, v_{i_{j+1}}, \ldots, v_{i_k}\right]$, for all $2 \leq j \leq k$ and for all $i_2 \leq \ldots \leq i_j$, $i_{j+1} \leq \ldots \leq i_k$,
- $[v_{i_1}, v_{i_2}, \ldots, v_{i_k}]$, for all $i_1 > i_2 \leq \ldots \leq i_k$,

where $k \geq 2$ and $0 < i_1, i_2, \ldots, i_k < p$.

Let A denote the subalgebra generated by $a_1, \ldots, a_{p-1}$. Similarly, denote by V the subalgebra generated by $v_1, \ldots, v_{p-1}$, and by I_V the ideal generated by V. Consider the quotient algebra $\tilde{L_p} = M(X_p)/[I_V, I_V]$. Since I_V is a homogeneous ideal, this Lie algebra inherits from $M(X_p)$ its $\mathbb{Z}$-grading. The generators form a basis for its homogeneous component of weight 1. For every $k \geq 2$, a basis for its homogeneous component of weight k is given by the Lie brackets

- $[a_{i_1}, a_{i_2}, \ldots, a_{i_k}]$, for all $i_1 > i_2 \leq \ldots \leq i_k$,
- $\left[a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}, v_{i_k}\right]$, for all $i_1 > i_2 \leq \ldots \leq i_{k-1}$,

- $[v_{i_1}, a_{i_2}, a_{i_3} \ldots, a_{i_k}]$, for all $i_2 \leq \ldots \leq i_k$,

where $0 < i_1, i_2, \ldots, i_k < p$. Indeed, it is not hard to prove that $[I_V, I_V]$ is generated by the remaining Lie brackets, which are the ones with more than one entry in V.

Finally, define the Lie algebra L_p to be $\tilde{L_p}/_{id}\langle[A, A]\rangle$. Similarly, since the ideal generated by the derived algebra of A is homogeneous, the algebra L_p is $\mathbb{Z}$-graded. A basis for it consists of X_p together with the Lie brackets

$$\mathcal{B}_{p,k} = \left\{[v_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}] \,:\, 0 < i_0 < p,\, 0 < i_1 \leq \ldots \leq i_{k-1} < p\right\}, \qquad (4.2)$$

for every $k \geq 2$. By construction, it follows that the Lie algebra L_p is a semidirect sum of the abelian ideal I_V and the abelian subalgebra A.

In the next proposition, we determine the dimension of the homogeneous components of L_p.

Proposition 4.4. *For every prime p and natural number k, let $l_{p,k}$ denote the dimension of $L_{p,k}$, the homogeneous component of L_p of weight k. We have*

$$l_{p,1} = 2(p-1) \qquad \text{and} \qquad l_{p,k} = (p-1)\binom{k+p-3}{k-1} \text{ for all } k \geq 2.$$

Proof. Since the set X_p is a basis for the subspace $L_{p,1}$, the first equality trivially holds. Now assume $k \geq 2$. By Formula (4.2), all possible Lie brackets in $\mathcal{B}_{p,k}$ with a fixed initial entry v_{i_0} are determined by the $(k-1)$-combinations with repetition from $p-1$ elements. The expression for $l_{p,k}$ follows from the fact that the number of these combinations is exactly $\binom{(k-1)+(p-1)-1}{k-1}$ and v_{i_0} varies among $p-1$ elements. □

The Lie algebra L_p admits a Frobenius group of automorphisms $C_p \rtimes C_{p-1}$, also denoted simply by C_pC_{p-1}, where C_p and C_{p-1} are cyclic groups of order respectively p and $p-1$. Let f denote a generator of the Frobenius kernel C_p. For each $i = 1, \ldots, p-1$, we define $a_i^f = \omega^i a_i$ and $v_i^f = \omega^i v_i$, where ω is a complex primitive pth root of unity. Let h be a generator of the Frobenius complement C_{p-1} and let r be a primitive $(p-1)$th root of unity in $\mathbb{Z}/p\mathbb{Z}$. We define $a_i^h = a_{ri}$ and $v_i^h = v_{ri}$, where the indices on the right-hand side must be taken modulo p. It is straightforward to check that these conditions determine a Frobenius group where $f^{h^{-1}} = f^r$.

Let $C_{L_p}(C_p)$ and $C_{L_p}(C_{p-1})$ respectively denote the centralisers of the Frobenius kernel and the Frobenius complement. They are both $\mathbb{Z}$-graded subspaces, since the homogeneous components $L_{p,k}$ are C_pC_{p-1}-invariant. A basis for $C_{L_p}(C_p) \cap L_{p,k}$ is obtained by those Lie brackets $[v_{i_0}, a_{i_1}, \dots, a_{i_{k-1}}]$ in $\mathcal{B}_{p,k}$ satisfying the condition $i_0 + i_1 + \cdots + i_{k-1} \equiv 0 \pmod{p}$. The next proposition presents a basis for the subspace $C_{L_p}(C_{p-1}) \cap L_{p,k}$.

Proposition 4.5. *The basis $\mathcal{B}_{p,k}$ of $L_{p,k}$ is permuted by C_{p-1}. The number of its orbits is equal to the dimension of the subspace $C_{L_p}(C_{p-1}) \cap L_{p,k}$, namely $l_{p,k}/(p-1)$. Indeed, the set $\left\{\sum_{j=0}^{p-2} b^{h^j},\ b \in \mathcal{B}_{p,k}\right\}$ is a basis for such subspace.*

Proof. First we need to prove that the generator h of C_{p-1} acts as a permutation of $\mathcal{B}_{p,k}$. This follows directly from the formula for $\mathcal{B}_{p,k}$. Indeed, for every Lie bracket $[v_{i_0}, a_{i_1}, \dots, a_{i_{k-1}}]$ in this set, its image under h is $[v_{ri_0}, a_{ri_1}, \dots, a_{ri_{k-1}}]$, again contained in $\mathcal{B}_{p,k}$, possibly after a rearrangement of the terms a_{ri_j}. Furthermore, every orbit has length $p-1$. Indeed, for $j = 0, \dots, p-2$, the images $[v_{r^j i_0}, a_{r^j i_1}, \dots, a_{r^j i_{k-1}}]$ under h^j are all distinct because they differ in the first entry. As a consequence, we get that the number of orbits is $|\mathcal{B}_{p,k}|/(p-1)$.

To each orbit there corresponds one vector in $C_{L_p}(C_{p-1}) \cap L_{p,k}$, namely the sum over its elements. This follows directly from the structure of a permutation module, where to each cycle of the permutation there corresponds one eigenvector with eigenvalue 1. $\square$

The Lie algebra L_p also has a natural $(\mathbb{Z}/p\mathbb{Z})$-grading, arising from the eigenspace decomposition corresponding to the action of C_p. Indeed, for every $0 \leq j < p$, define ${}^jL_p = \left\{x \in L_p | x^f = \omega^j x\right\}$. We call this subspace the C_p-homogeneous component of degree j.

Because of the definition of f, a Lie bracket $[v_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}]$ in $\mathcal{B}_{p,k}$ belongs to such subspace if and only if $i_0 + i_1 + \cdots + i_{k-1} \equiv j \pmod p$. Clearly, the C_p-homogeneous component of degree 0 corresponds to the centraliser $C_{L_p}(C_p)$ and it is C_{p-1}-invariant.

Since the homogeneous components $L_{p,k}$ are C_p-invariant, they are, in turn, $(\mathbb{Z}/p\mathbb{Z})$-graded by this basic fact of linear algebra:

Lemma 4.6. *Let n be a positive integer and let ω be a primitive nth root of unity. Let K be a field of characteristic coprime to n and containing ω. Let V be a K-vector space. Let $\phi : V \to V$ be a linear map of order n and U a ϕ-invariant subspace of V. Assume that an element u of U can be written in the form $u = u_0 + \cdots + u_{n-1}$, where u_i are eigenvectors of ϕ corresponding to distinct eigenvalues ω^i. Then $u_i \in U$ for all $0 \leq i \leq n-1$.*

Proof. Since U is ϕ-invariant, $u^{\phi^j} \in U$ for all $j = 0, \ldots, n-1$. Writing them explicitly we have

$$\begin{array}{rrrr}
u_0 + & u_1 + & u_2 + \cdots + & u_{n-1} \in U \\
u_0 + & \omega u_1 + & \omega^2 u_2 + \cdots + & \omega^{n-1} u_{n-1} \in U \\
u_0 + & \omega^2 u_1 + & \omega^4 u_2 + \cdots + & \omega^{2(n-1)} u_{n-1} \in U \\
\cdots & & & \\
\multicolumn{4}{l}{u_0 + \omega^{(n-1)} u_1 + \omega^{2(n-1)} u_2 + \cdots + \omega^{(n-1)(n-1)} u_{n-1} \in U.}
\end{array}$$

To prove that $u_i \in U$, we multiply the jth row by $\omega^{-(j-1)i}$ and sum up. The result will be trivially contained in U. This is exactly nu_i, since for all other terms u_k the coefficients are $\sum_{j=1}^{n} \omega^{(j-1)(k-i)} = 0$. The conclusion follows dividing by n. $\square$

Hence, we can write

$$L_{p,k} = \bigoplus_{j=0}^{p-1} L_{p,k} \cap {}^{j}L_p.$$

The subspaces on the right-hand side are permuted by h, according to the formula $(L_{p,k} \cap {}^{j}L_p)^h = L_{p,k} \cap {}^{rj}L_p$. Hence, for every prime p and integer k, the subspaces $W_{p,k} = L_{p,k} \cap {}^{0}L_p$ and $W'_{p,k} = L_{p,k} \cap {}^{1}L_p \oplus \cdots \oplus L_{p,k} \cap {}^{p-1}L_p$ are C_pC_{p-1}-modules. The basis of $W'_{p,k}$ consists of eigenvectors for the action of C_p which are permuted by C_{p-1} in orbits of length $p-1$. Each of these orbits spans an irreducible C_pC_{p-1}-module of dimension $p-1$ which is induced by an irreducible (one-dimensional) module of C_p (Theorem 1.7). Hence, we have that

$$\dim(W'_{p,k}) = (p-1)\dim(C_{L_p}(C_{p-1}) \cap W'_{p,k}).$$

Using Proposition 4.5 and the C_{p-1}-invariance of both subspaces, we get the corresponding formula for $W_{p,k}$.

4.2 The ideal generated by $C_{L_p}(C_p)$

Let I_p denote the ideal generated by the centraliser $C_{L_p}(C_p)$. We have already mentioned that $C_{L_p}(C_p)$ is weight-homogeneous and C_pC_{p-1}-invariant. Hence, the same properties hold for I_p. In this section, for every natural number k, we bound the dimension of the subspaces $I_p \cap L_{p,k}$.

Since v_i and a_i are eigenvectors of f with corresponding eigenvalues ω^i, a basis for the subspace $I_p \cap L_{p,k}$ is given by those Lie brackets $[v_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}]$ belonging to $\mathcal{B}_{p,k}$ for which there exists a permutation π in the symmetric group S_{k-1} such that $i_0 + i_{\pi(1)} + \cdots + i_{\pi(s)} \equiv 0 \pmod p$, for some $s < k$. In the next proposition, we bound the dimension of this subspace.

Proposition 4.7. *For every prime number p and natural number k, let $i_{p,k}$ denote the dimension of the homogeneous subspace $I_p \cap L_{p,k}$. Then*

$$i_{p,1} = 0 \qquad \textit{and} \qquad i_{p,k} \leq (k-1)(p-1)^{k-1} \quad \textit{for all } k \geq 2.$$

Proof. First, we recall that $L_{p,1}$ is the subspace of homogeneous elements of L_p of weight 1 and it is generated by X_p. Since none of the v_i, a_i is centralised by C_p, we clearly have $i_{p,1} = 0$. More generally, we have

$$\begin{aligned} I_p \cap L_{p,k} &= \sum_{j=1}^{k} [C_{L_p}(C_p) \cap L_{p,j}, \underbrace{L_{p,1}, \ldots, L_{p,1}}_{k-j}] \\ &= \sum_{j=1}^{k-1} [[C_{L_p}(C_p) \cap L_{p,j}, \underbrace{L_{p,1}, \ldots, L_{p,1}}_{k-1-j}], L_{p,1}] + C_{L_p}(C_p) \cap L_{p,k} \\ &= \Big[\sum_{j=1}^{k-1} [C_{L_p}(C_p) \cap L_{p,j}, \underbrace{L_{p,1}, \ldots, L_{p,1}}_{k-1-j}], L_{p,1} \Big] + C_{L_p}(C_p) \cap L_{p,k} \\ &= [I_p \cap L_{p,k-1}, L_{p,1}] + C_{L_p}(C_p) \cap L_{p,k}. \end{aligned}$$

Hence, we obtain $i_{p,k} \leq \dim([I_p \cap L_{p,k-1}, L_{p,1}]) + \dim(C_{L_p}(C_p) \cap L_{p,k})$.

The subspace $C_{L_p}(C_p) \cap L_{p,k}$ has basis given by the homogeneous Lie brackets $[v_{i_0}, a_{i_1}, \ldots, a_{i_{k-1}}]$ in $\mathcal{B}_{p,k}$ for which $i_0 + i_1 + \cdots + i_{k-1} \equiv 0 \pmod p$. This implies that the last index i_{k-1} is uniquely determined by the previous ones and hence, $\dim(C_{L_p}(C_p) \cap L_{p,k}) \leq (p-1)^{k-1}$. Notice that this bound is only sharp for $k = 2$, because the set $\{[v_i, a_{p-i}],\ i = 1, \ldots, p-1\}$ is a basis for $I_p \cap L_{p,2}$. However, for $k \geq 3$ this argument counts some basis elements in $\mathcal{B}_{p,k}$ several times, as we consider all simple Lie bracket of weight k, not only the "ordered" ones.

The subspace $[I_p \cap L_{p,k-1}, L_{p,1}]$ is generated by the Lie brackets of the form $[x, a_j]$, where x ranges over a basis of $I_p \cap L_{p,k-1}$ and $j = 1, \ldots, p-1$. Hence, we have $\dim([I_p \cap L_{p,k-1}, L_{p,1}]) \leq (p-1) i_{p,k-1}$. Once again, we point out that the equality trivially holds for $k = 2$, and for $k = 3$. The inequality is instead strict for larger values of k. These bounds together give the recursive formula

$$i_{p,k} \leq (p-1) i_{p,k-1} + (p-1)^{k-1},$$

from which the conclusion follows by induction. $\square$

This proposition concludes this section and the study of the ideal I_p. Indeed, the bound on the dimension of its homogeneous components $I_p \cap L_{p,k}$ is all we need in the proof of Proposition 4.3.

4.3 The ideal generated by $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$

Let J_p denote the smallest C_pC_{p-1}-invariant ideal containing the derived subalgebra $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$. Since $C_{L_p}(C_{p-1})$ is not C_p-invariant, then J_p is strictly bigger than the ideal generated simply by $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$. Indeed, in this section we will prove that J_p is generated, as an ideal, by the C_p-homogeneous components of the generators of $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$. This is done in Proposition 4.8. Moreover, as previously done for I_p, we will bound the dimension of the subspaces $J_p \cap L_{p,k}$, for every prime number p and integer k (Proposition 4.9). The bound is obtained as the exact result of a recursion.

We first determine the generators of the derived subalgebra of $C_{L_p}(C_{p-1})$. As already mentioned, the centraliser $C_{L_p}(C_{p-1})$ is weight-homogeneous and a basis for the subspaces $C_{L_p}(C_{p-1}) \cap L_{p,k}$ is given in Proposition 4.5. The subalgebra $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ is generated by Lie brackets of the form $[x, y]$, where x and y lie in the basis of $C_{L_p}(C_{p-1})$. Those Lie brackets are trivial unless one of the entries is $a_1 + a_2 + \cdots + a_{p-1}$. Hence, the derived subalgebra of $C_{L_p}(C_{p-1})$ is generated by the Lie brackets of the form $[x, a_1 + \cdots + a_{p-1}]$, where x lies in the basis of $C_{L_p}(C_{p-1})$.

In the next proposition we produce a set of generators of the ideal J_p.

Proposition 4.8. *The smallest ideal containing $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ and invariant under the action of the group C_pC_{p-1} is generated, as an ideal, by the C_p-homogeneous components of the generators of $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$.*

Proof. Consider $[x, \sum_{i=1}^{p-1} a_i]$, where x belongs to the basis of $C_{L_p}(C_{p-1})$ given in Proposition 4.5. We can write x as a linear combination of eigenvectors of f, namely $x = \sum_{j=0}^{p-1} x_j$, where x_j is in the C_p-homogeneous component of degree j. Since x is in $C_{L_p}(C_{p-1})$, we necessarily have that $x_{rj} = (x_j)^h$, for all $j = 0, \dots p-1$. The same relation is true for the C_p-homogeneous components of $[x, \sum_{i=1}^{p-1} a_i]$. Indeed, plugging the decomposition of x into the Lie bracket, we can rewrite it as $\sum_{j=0}^{p-1} \left(\sum_{k \neq j} [x_k, a_{j-k}] \right)$, where the sum in parenthesis represents its C_p-homogeneous component of degree j. Applying h to it, we get

$$\left(\sum_{k \neq j} [x_k, a_{j-k}] \right)^h = \sum_{k \neq j} [x_k^h, a_{j-k}^h] = \sum_{k \neq j} [x_{rk}, a_{rj-rk}],$$

where this should be interpreted as a sum over k, where k varies from 0 to $p-1$. This is exactly the homogeneous component of degree rj.

We have just shown that the set of the C_p-homogeneous components of the generators of $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ is permuted by C_{p-1}. This implies that the ideal generated by this set is C_{p-1}-invariant. Moreover, this ideal is also C_p-invariant because, by construction, its generators are C_p-homogeneous. It contains the subalgebra $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ because it contains all its generators (as a sum of their C_p-homogeneous components). Hence, it contains J_p, for minimality of J_p.

For the opposite inclusion, it is sufficient to show that all C_p-homogeneous components of the generators of $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ are in J_p. This is true by Lemma 4.6. □

We can now bound the dimension of the weight-homogeneous components of J_p.

Proposition 4.9. *For every prime number p and natural number k, let $j_{p,k}$ denote the dimension of the subspace $J_p \cap L_{p,k}$, the homogeneous component of J_p of weight k. Then*

$$j_{p,1} = 0 \qquad \textit{and} \qquad j_{p,k} \leq p \left(\sum_{j=0}^{k-2} (p-1)^{k-2-j} \binom{j+p-2}{j} \right) \textit{ for all } k \geq 2.$$

Proof. First, observe that the ideal J_p is contained in the derived subalgebra of L_p. This implies trivial intersection with the subspace $L_{p,1}$ of homogeneous elements of weight 1 and hence $j_{p,1} = 0$.

More generally, if x ranges in the set of generators of the subalgebra $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})]$ and x_l denotes its C_p-homogeneous component of degree l, by the previous result we can write

$$\begin{aligned}
J_p \cap L_{p,k} &= \sum_{j=1}^{k}\Big(\sum_{l=0}^{p-1}\big[\langle x_l : x \in [C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,j}\rangle, \underbrace{L_{p,1}, \dots, L_{p,1}}_{k-j}\big]\Big)\\
&= \sum_{j=1}^{k-1}\Big(\sum_{l=0}^{p-1}\big[\big[\langle x_l : x \in [C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,j}\rangle, \underbrace{L_{p,1}, \dots, L_{p,1}}_{k-1-j}\big], L_{p,1}\big]\Big)\\
&\quad + \sum_{l=0}^{p-1}\langle x_l : x \in [C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,k}\rangle\\
&= [J_p \cap L_{p,k-1}, L_{p,1}] + \sum_{l=0}^{p-1}\langle x_l : x \in [C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,k}\rangle.
\end{aligned}$$

Hence, we have

$$j_{p,k} \leq \dim([J_p \cap L_{p,k-1}, L_{p,1}]) + p\dim([C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,k}).$$

The subalgebra $[J_p \cap L_{p,k-1}, L_{p,1}]$, which is non-trivial only when $k \geq 3$, is generated by the Lie brackets of the form $[x, a_j]$, where x is an element of the basis of $J_p \cap L_{p,k-1}$ and $j = 1, \dots, p-1$. Indeed, the Lie brackets $[x, v_i]$ are trivial by construction. Hence, we get $\dim([J_p \cap L_{p,k-1}, L_{p,1}]) \leq (p-1)j_{p,k-1}$.

As previously remarked, the subalgebra $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,k}$ is generated by the Lie brackets $[x, \sum_{i=1}^{p-1} a_i]$, where x is in the basis of $C_{L_p}(C_{p-1}) \cap L_{p,k-1}$. Hence, its dimension is not greater than the dimension of $C_{L_p}(C_{p-1}) \cap L_{p,k-1}$, namely $l_{p,k-1}/(p-1)$, by Proposition 4.5. Using the expression for $l_{p,k}$ computed in Proposition 4.4, we get the following recursive formula

$$j_{p,k} \leq (p-1)j_{p,k-1} + p\binom{k+p-4}{k-2}.$$

We now prove the desired bound for $j_{p,k}$ by induction on k. Observe that the subspace $[C_{L_p}(C_{p-1}), C_{L_p}(C_{p-1})] \cap L_{p,2}$ is one-dimensional, generated by $[\sum_{i=1}^{p-1} v_i, \sum_{j=1}^{p-1} a_j]$. According to Proposition 4.8, its C_p-homogeneous components generate $J_p \cap L_{p,2}$. Hence, the bound holds for $k = 2$. Suppose now that the bound is true for $k-1$. Using the above inequality in the inductive hypothesis, we find:

$$\begin{aligned} j_{p,k} &\leq (p-1)p\left(\sum_{j=0}^{k-3}(p-1)^{k-3-j}\binom{j+p-2}{j}\right) + p\binom{k+p-4}{k-2} \\ &= p\left(\sum_{j=0}^{k-3}(p-1)^{k-2-j}\binom{j+p-2}{j}\right) + p\binom{k+p-4}{k-2} \\ &= p\left(\sum_{j=0}^{k-2}(p-1)^{k-2-j}\binom{j+p-2}{j}\right). \end{aligned}$$

This concludes the proof. □

This also concludes the section about the ideal J_p. As in the case of the ideal I_p, all we will need later on is the bound for the dimension of its homogeneous components.

4.4 Proofs of Proposition 4.3 and Theorem 4.2

In this last section, we will prove Proposition 4.3 by explicit construction of the Lie algebras $\bar{L}_p$ belonging to the family $\mathfrak{L}$. We will then apply Lazard correspondence to prove Theorem 4.2, where we will need to impose some conditions on the underlying field of each $\bar{L}_p$ in $\mathfrak{L}$. Recall that, until now, the only requirements on the underlying field of L_p were having characteristic different from p and containing a primitive pth root of unity.

Proof of Proposition 4.3. For every prime number p, define $\bar{L}_p$ to be the quotient algebra $L_p/(I_p+J_p)$. Since the two ideals I_p and J_p are graded, the same is true for their sum. In particular, every homogeneous component is given by the sum of the corresponding ones: $(I_p + J_p) \cap L_{p,k} = I_p \cap L_{p,k} + J_p \cap L_{p,k}$. This implies that $\bar{L}_p$ inherits from L_p its $\mathbb{Z}$-grading. Let $\bar{L}_{p,k}$ denote the image of $L_{p,k}$ in the quotient and let $\bar{l}_{p,k}$ indicate its dimension. We have

$$\begin{aligned} \bar{l}_{p,k} &= l_{p,k} - \dim((I_p + J_p) \cap L_{p,k}) = l_{p,k} - \dim(I_p \cap L_{p,k} + J_p \cap L_{p,k}) \\ &\geq l_{p,k} - \dim(I_p \cap L_{p,k}) - \dim(J_p \cap L_{p,k}) = l_{p,k} - i_{p,k} - j_{p,k}. \end{aligned}$$

Substituting in this formula the expression for $l_{p,k}$ from Proposition 4.4, and the upper bounds for $i_{p,k}$ and $j_{p,k}$, respectively from Proposition 4.7 and 4.9, we get

$$\bar{l}_{p,k} \geq (p-1)\binom{k+p-3}{k-1} - (k-1)(p-1)^{k-1} - p\left(\sum_{j=0}^{k-2}(p-1)^{k-2-j}\binom{j+p-2}{j}\right).$$

We observe that the right-hand side is a polynomial function of the prime number p, whose leading term is $p^k/(p-1)!$ and comes from the formula for $l_{p,k}$.

The algebra $\bar{L}_p$ admits the same Frobenius group C_pC_{p-1}, as a group of automorphisms. Indeed, since I_p and J_p are C_pC_{p-1}-invariant, the generator f of C_p and the generator h of C_{p-1} induce automorphisms of the quotient algebra, defined in the canonical way. The orders of f and h remain unchanged in their action on the quotient, namely equal to p and $p-1$ respectively.

The fixed-point subspace $C_{\bar{L}_p}(C_p)$ is trivial. Indeed, in case of a coprime action, the fixed points in the quotient algebra are covered by the fixed points in the original Lie algebra (Theorem 1.17). For the same reason, if we require that, for every prime p, the characteristic of the underlying field of $\bar{L}_p$ is coprime with $p-1$, then the subspace $[C_{\bar{L}_p}(C_{p-1}), C_{\bar{L}_p}(C_{p-1})]$ is trivial too. At the end of this section we will show that this hypothesis is unnecessary. Indeed, the fixed points of C_{p-1} in the quotient algebra are covered by the fixed points in the original Lie algebra even when the characteristic of the field divides $p-1$. Nevertheless, for now just assume that this condition holds, so that the three properties of $\mathfrak{L}$ listed in Proposition 4.3 are satisfied and we can continue with our proof.

It only remains to show that for every integer k, there exists a Lie algebra in $\mathfrak{L}$ with nilpotency class at least k. In other words, this algebra must have a non-trivial Lie bracket of weight k. Because the lower bound for $\bar{l}_{p,k}$ has positive leading coefficient, for every fixed k we can always find a prime number p for which $\bar{l}_{p,k}$ is positive. Hence, the corresponding Lie algebra will have nilpotency class at least k. Proposition 4.3 is now proved modulo Lemma 4.10. $\square$

We are now ready to prove the main result of this chapter.

Proof of Theorem 4.2. Every $\bar{L}_p$ is a metabelian $(\mathbb{Z}/p\mathbb{Z})$-graded Lie algebras with ${}^0\bar{L}_p = 0$. According to a theorem of Shumyatsky [Shu05, Theorem 3.3], its nilpotency class is at most $p-1$. We can therefore produce a similar result for groups under the extra condition that, for every algebra $\bar{L}_p$ (and hence L_p), the characteristic q_p of the underlying field is greater than p. Indeed, under this hypothesis, the Lazard correspondence based on the 'truncated' Baker–Campbell–Hausdorff formula transforms every $\bar{L}_p$ into a finite q_p-group Q_p with the same nilpotency class as $\bar{L}_p$. The group Q_p admits the same Frobenius group of automorphisms C_pC_{p-1} with trivial $C_{Q_p}(C_p)$ and abelian $C_{Q_p}(C_{p-1})$. This proves Theorem 4.2. $\square$

Increasing the characteristic of the field with p in our construction was only required for an application of the Lazard correspondence. Hence, it is not unnatural to ask the following

Question. We wonder if it is possible to construct a family of q-groups, of unbounded nilpotency class, where q is a fixed prime number, whose elements G satisfy:

1. G admits a metacyclic Frobenius group of automorphisms;

2. the centraliser of the Frobenius kernel in G is trivial;

3. the centraliser of the Frobenius complement in G is abelian.

We conclude this section by proving that the fixed points of C_{p-1} in $\bar{L}_p$ are covered by the fixed points of the same group in L_p even when the action is not coprime.

Lemma 4.10. *For every prime number p let L_p, $\bar{L}_p$, I_p and J_p defined as above. Let T_p denote the ideal $I_p + J_p$, so that $\bar{L}_p = L_p/T_p$. Then we have*

$$C_{\bar{L}_p}(C_{p-1}) = (C_{L_p}(C_{p-1}) + T_p)/T_p.$$

Proof. We can restrict ourselves to the weight-homogeneous case and, denoting the subspace $(I_p + J_p) \cap L_{p,k}$ by $T_{p,k}$, show that

$$C_{L_{p,k}/T_{p,k}}(C_{p-1}) = (C_{L_{p,k}}(C_{p-1}) + T_{p,k})/T_{p,k}.$$

Recall that $W_{p,k} = L_{p,k} \cap {}^0L_p$ and $W'_{p,k} = L_{p,k} \cap {}^1L_p \oplus \cdots \oplus L_{p,k} \cap {}^{p-1}L_p$. Because I_p and J_p are C_p-invariant we have

$$T_{p,k} = (I_p + J_p) \cap W_{p,k} \oplus (I_p + J_p) \cap W'_{p,k} = W_{p,k} \oplus (I_p + J_p) \cap W'_{p,k},$$

whence

$$\frac{L_{p,k}}{T_{p,k}} \cong \frac{W'_{p,k}}{(I_p + J_p) \cap W'_{p,k}}.$$

A set of generators for $(I_p + J_p) \cap W'_{p,k}$ is given by the generators of $I_p \cap W'_{p,k}$ together with the generators of $J_p \cap W'_{p,k}$. In other words, since the generators of $(I_p + J_p) \cap L_{p,k}$, described in Propositions 4.7 and 4.8, are all C_p-homogeneous, we take those of them not belonging to ${}^0L_p$.

It is not hard to show that the generators of $I_p \cap W'_{p,k}$ are permuted by C_{p-1} in orbits of length $p - 1$, whose elements are all in different C_p-homogeneous components (one for each ${}^jL_p$, with $1 \le j \le p-1$). The same holds for the generators of $J_p \cap W'_{p,k}$. The key point here is that each simple

Lie bracket involved in one of those generators has sum of its indices different from 0 modulo p. We have just shown that

$$(I_p + J_p) \cap W'_{p,k} = \bigoplus_{j=0}^{p-2} ((I_p + J_p) \cap L_{p,k} \cap {}^1L_p)^{h^j}.$$

Now consider a basis $\{w_1, \ldots, w_t\}$ for the subspace $(I_p + J_p) \cap L_{p,k} \cap {}^1L_p$. From the above equation it follows that $(I_p + J_p) \cap W'_{p,k}$ has basis

$$\left\{ w_1, \ldots, w_t, w_1^h, \ldots, w_t^h, \ldots, w_1^{h^{p-2}}, \ldots, w_t^{h^{p-2}} \right\}.$$

If we complete the basis of $(I_p + J_p) \cap L_{p,k} \cap {}^1L_p$ to a basis of $L_{p,k} \cap {}^1L_p$, by adding the vectors $\{w_{t+1}, \ldots, w_s\}$, then a basis of $L_{p,k}/(I_p + J_p) \cap L_{p,k}$ is

$$\left\{ w_{t+1}, \ldots, w_s, w_{t+1}^h, \ldots, w_s^h, \ldots, w_{t+1}^{h^{p-2}}, \ldots, w_s^{h^{p-2}} \right\},$$

where, with a little abuse of notation, the cosets are named by their representative. The desired conclusion follows. $\square$

We remark that the result of the previous lemma strongly relies on the definitions of the algebras involved. Indeed, a crucial step in the proof is the fact that both L_p and $\bar{L}_p$ are free H-modules.

Chapter 5

Graded Lie algebras of maximal class of type p

This chapter can be read independently of the rest of the thesis. Indeed, we present a completely different problem: the study and classification of graded Lie algebras of maximal class of type p. These Lie algebras have been considered for the first time by Claudio Scarbolo in his PhD thesis [Sca14], to which we refer for most of the results presented here.

In the first section we introduce these Lie algebras. In the second section we give the definitions of two-step centralisers and constituents. We will also recall some binomial identities that will be used later on. The last three sections are devoted to our contribution to their classification. It consists of a different proof of a proposition regarding the length of the first constituent, which will be based on some stand-alone results about polynomials.

5.1 Graded Lie algebras of maximal class

The concept of coclass was first introduced by Leedham-Green and Newman in the context of p-groups and pro-p groups as a measure of closeness to being of maximal class ([LGN80]). Analogously, one can define the coclass of a finite-dimensional nilpotent Lie algebra L as $cc(L) = n - c$, where n is the dimension of L and c is its nilpotency class. This definition naturally extends to the infinite-dimensional case. For a finitely generated residually nilpotent Lie algebra L, we set

$$cc(L) = \sum_{\substack{i>0 \\ \gamma_i(L)\neq 0}} (\dim(\gamma_i(L)/\gamma_{i+1}(L)) - 1),$$

where $\gamma_i(L)$ represents the ith term of its lower central series (in the literature generally denoted by L^i). When the coclass is minimal, namely $cc(L) = 1$, we say that L is of maximal class. Equivalently, a Lie algebra of maximal class is a residually nilpotent Lie algebra L such that $\dim(L/L^2) = 2$ and $\dim(L^i/L^{i+1}) \leq 1$ for all $i > 1$.

Vergne showed that the family of Lie algebras of maximal class is already very big in characteristic zero ([Ver70]). For this reason, many authors have focused their attention to the $\mathbb{N}$-graded ones, that is of the form

$$L = \bigoplus_{i=1}^{\infty} L_i, \qquad \text{with } [L_i, L_j] \subseteq L_{i+j}.$$

More precisely, L is a $\mathbb{Z}$-graded Lie algebra where $L_i = 0$, for every $i \leq 0$. In this chapter, when talking about Lie algebras of maximal class without further specifications, we will implicitly mean the $\mathbb{N}$-graded ones.

Shalev and Zelmanov studied ($\mathbb{N}$-graded) Lie algebras of maximal class generated by the first homogeneous component L_1. These have been called Lie algebras of type 1. The study of this specific situation is natural and comes from the study of the groups of maximal class. The authors proved that in characteristic zero there is only one just infinite algebra, which is metabelian ([SZ97]).

Caranti, Mattarei and Newman focused their attention on Lie algebras of maximal class of type 1 defined over fields of positive characteristic. This situation is far more complex. Indeed, the authors proved that over every field F of positive characteristic one can have $|F|^{\aleph_0}$ infinite-dimensional non-isomorphic Lie algebras of maximal class ([CMN97]). If the characteristic of the field is odd, then they are all constructed from simple finite-dimensional Lie algebras described by Albert and Frank, using a method called inflation, possibly infinitely many times, or as limits of families of finite-dimensional quotients of known algebras ([CN00]). The case of characteristic 2 is essentially more complicated and it was studied by Jurman in [Jur05].

Lie algebras of type 1 do not exhaust all possibilities for graded Lie algebras of maximal class. For instance, one can consider those Lie algebras generated by the first and second homogeneous components, called Lie algebras of type 2, where all homogeneous components have dimension at most 1. This is done in [SZ97] for characteristic zero and in [CVL00] for positive characteristic, finally in [CVL03] for characteristic 2.

In [Ugo10] the author considers the case of graded Lie algebras of maximal class of type n, namely generated by the first and the nth homogeneous component, over fields of characteristic greater than $2n$.

In this chapter we consider graded Lie algebras L of maximal class of type p over fields of characteristic p. Specifically, these Lie algebras are generated by an element of degree 1 and an element of degree a prime number p:

$$L = L_1 \oplus \bigoplus_{i \geq p} L_i,$$

and every homogeneous component has dimension 1. Since the case $p = 2$ was covered by Caranti and Vaughan-Lee in [CVL00, CVL03], we will look at infinite-dimensional Lie algebras defined over fields of odd characteristic. Hence, even when not specifically mentioned, F will denote a field with odd characteristic p. We will use the symbol $L_{(i)}$ to denote the ideal of L given by $\bigoplus_{j \geq i} L_j$. Note that, in cases of graded Lie algebras of maximal class, $L_{(i)} = L^i$ holds only for Lie algebras of type 1.

In his PhD Thesis, Scarbolo obtained a classification of these Lie algebras.

Theorem 5.1. *Let F be a field of odd characteristic p and let L be a graded Lie algebra of maximal class of type p over F such that $[L^2, L^2] \subseteq L_{(4p+2)}$. Then, either L is obtained by translating a subalgebra of an uncovered Lie algebra of type 1 or a L belongs to a finite family $\mathcal{E}$.*

Note that $L^2 = L_{(p+1)}$ and the ordinary inclusion is $[L^2, L^2] \subseteq L_{(2p+3)}$. The hypothesis $[L^2, L^2] \subseteq L_{(4p+2)}$ was not present in the original version of this theorem in [Sca14, Theorem 2.6]. In language which we introduce in the next section, this hypothesis amounts to the length ℓ of the first constituent being strictly larger than $4p$. In its absence Scarbolo's proof turns out to be incomplete.

The finite family $\mathcal{E}$ of the "exceptional" Lie algebras is explicitly constructed in Scarbolo's thesis, to which we refer the interested reader. The notion of uncovered Lie algebra and explanations about translations are given below. We stress that this result can be seen as a generalisation of the classification theorem for graded Lie algebras of maximal class of type 2 in characteristic 2. In fact, in this case the classification theorem states that every such Lie algebra is obtained as a translation of a subalgebra of an uncovered Lie algebra of type 1 (see [CVL03]).

5.2 Two-step centralisers and constituents of Lie algebras of type p

In this section we introduce the concepts of two-step centralisers and of constituents for Lie algebras of maximal class of type p. The original definition of two-step centralisers for Lie algebras of type 1 was motivated by the corresponding one for p-groups of maximal class. Indeed, they are defined in [CMN97] as $C_L\left(L^i/L^{i+2}\right)$. This corresponds to the centraliser in L_1 of the homogeneous component L_i for all $i > 2$, as $C_L(L^i/L^{i+2}) = C_{L_1}(L_i) + L^2$.

This definition, applied to an uncovered Lie algebra (see Example 5.2 for the relevant definition) has then inspired that of two-step centralisers for Lie algebras of maximal class of different types.

Contrary to the definition of two-step centraliser, there have been inconsistencies in the definition of constituents. The definition presented here is inspired by the one given in [CVL03] for Lie algebras of type 2.

Let L be a Lie algebra of maximal class of type p over a field F of characteristic p, generated by e_1 of degree 1 and e_p of degree p. For all $i > p$, we have $L_i = Fe_i$, where $e_i = [e_p, e_1^{i-p}] = [e_p, \underbrace{e_1, \dots, e_1}_{i-p}]$. Note that, by definition, $L_i = 0$ for all $1 < i < p$. We define the sequence $\{\beta_i\}_{i>p}$ of two-step centralisers of L by

$$[e_i, e_p] = \beta_i e_{i+p}.$$

This sequence of elements of F encodes the adjoint action of e_p, while the action of e_1 is actually used to define e_i. It follows that the Lie algebra L is completely determined by this sequence, essentially because the adjoint action of the generators lets us know the adjoint action of every element of the Lie algebra.

This definition depends on the choice of the generators of L, which are unique up to scalar multiples. Indeed, if we consider another pair of generators $e_1' = \lambda e_1$ and $e_p' = \mu e_p$, for some $\lambda, \mu \in F^*$, then for all $i > p$ we have

$$e_i' = [e_{i-1}', e_1'] = \lambda^{i-p}\mu e_i \qquad \text{and} \qquad [e_i', e_p'] = \lambda^{i-p}\mu^2[e_i, e_p] = \frac{\mu}{\lambda^p}\beta_i e_{i+p}',$$

meaning that $\beta_i' = \frac{\mu}{\lambda^p}\beta_i$. It follows that one can scale the two-step centralisers of a Lie algebra of type p by a non-zero factor, without affecting the isomorphism type of the Lie algebra. Hence, from now on we will always work with the normalised sequence, that is the one where the first non-zero two-step centraliser (when it exists) equals 1. The next remark is then straightforward.

Remark 5.2. Let L and L' be two Lie algebras of type p over a field F of characteristic p, with (normalised) sequences of two-step centralisers respectively $\{\beta_i\}_{i>p}$ and $\{\beta_i'\}_{i>p}$. Then L and L' are isomorphic if and only if their sequences of two-step centralisers coincide. This follows from the fact that we are considering graded isomorphisms, that are of the form described above.

We now introduce the notion of constituents of a Lie algebra of type p, which is based on the sequence of two-step centralisers. A theory of constituents is always possible whenever $\beta_{p+1} = 0$, condition which is assumed to hold. Indeed, this hypothesis is not restrictive for the purpose of the classification, as shown in [Sca14]. We will discuss this concept more in detail later on.

Definition 5.3. Let L be a Lie algebra of type p, with sequence of two-step centralisers $\{\beta_i\}_{i>p}$. Suppose that $\beta_{p+1} = \cdots = \beta_{\ell-p} = 0$ and $\beta_{\ell-p+1} \neq 0$. Then we define the *first constituent* of L as

$$(\beta_{p+1}, \ldots, \beta_{\ell-p}, \beta_{\ell-p+1}, \ldots, \beta_{\ell}),$$

and we say that it has length ℓ. Observe that $\ell = \dim(L^2/[L^2, L^2]) + p$. This follows from the fact that $L^2 = \bigoplus_{j \geq p+1} L_j$ while $[L^2, L^2] = \bigoplus_{j \geq \ell+1} L_j$.

An exceptional situation is given by the metabelian Lie algebra M, whose two-step centralisers are all zero. In this case we say that $\ell = \infty$.

Note that if $[e_{p+1}, e_p] = 0$, then $[e_{\ell+1}, e_p] = 0$. Indeed, we have

$$0 = [e_{\ell-p}, [e_p, e_1, e_p]] = -[e_{\ell-p+1}, e_p, e_p] = \beta_{\ell-p+1}[e_{\ell+1}, e_p].$$

More generally, for every natural number n such that $\beta_{n-p} = 0$ and $\beta_{n-p+1} \neq 0$, we have $\beta_{n+1} = 0$. Hence, we can recursively define the other constituents.

Definition 5.4. Suppose that $(\beta_i, \ldots, \beta_k)$ is a constituent of a Lie algebra of maximal class of type p. If $\beta_{k+1} = \ldots = \beta_{k+m-p} = 0$ and $\beta_{k+m-p+1} \neq 0$, then

$$(\beta_{k+1}, \ldots, \beta_{k+m-p}, \beta_{k+m-p+1}, \ldots, \beta_{k+m})$$

is a *constituent* of length m.

The first constituent requires a special treatment, as in all other variants of this definitions. This convention allows for simpler statements of several results. A priori each of the last $p-1$ two-step centralisers of every constituent could be zero or non-zero. Figure 5.1 illustrates the first two constituents, of lengths respectively ℓ and h, of a generic Lie algebra of maximal class of type p. The dashed lines at the end of every constituent represent the missing information.

Note that the constituents, as defined here, are dubbed "fake" in Scarbolo's thesis. Indeed, he prefers to stop every constituent to the last non-zero two-step centraliser and then attach the remaining zero two-step centralisers to the beginning of the consecutive constituent. However, the definition given here appears to us as the most natural choice, as it is essential in all calculations.

The next example is very important for the purpose of the classification. It gives a characterisation of the Lie algebras of type p that are subalgebras of uncovered Lie algebras of type 1.

Example 5.5 (**Subalgebra of an uncovered Lie algebra of type** 1). Let $N = \bigoplus_{i\geq 1} N_i$ be an uncovered Lie algebra of maximal class of type 1, which means that there exists an element $e_1 \in N_1$ such that $N_i = [N_{i-1}, e_1]$ for all $i \geq 1$. For every $j \geq 2$ the centraliser $C_{N_1}(N_j)$ of N_j in N_1 is one-dimensional. Choose $y \in N_1 \setminus \langle e_1 \rangle$ so that $C_{N_1}(N_2) = \langle y \rangle$. We set the basis $e_2 = [y, e_1]$ and $e_i = [e_{i-1}, e_1]$. The corresponding sequence $\{\alpha_i\}_{i>p}$ of two-step centralisers is then defined by the formula $[e_i, y] = \alpha_i e_{i+1}$ for all $i > p$. This definition is equivalent to the one given in [CMN97] and recalled at the beginning of this chapter. Indeed, we have $C_{N_1}(N_i) = \langle y - \alpha_i e_1 \rangle$. Because of our choice for the two generators, we have $\alpha_2 = 0$.

Define L to be the subalgebra of N of the form $L = L_1 \oplus \bigoplus_{i\geq p} L_i$, where $L_1 = \langle e_1 \rangle$ and $L_i = N_i$, for all $i \geq p$. This Lie algebra is of maximal class

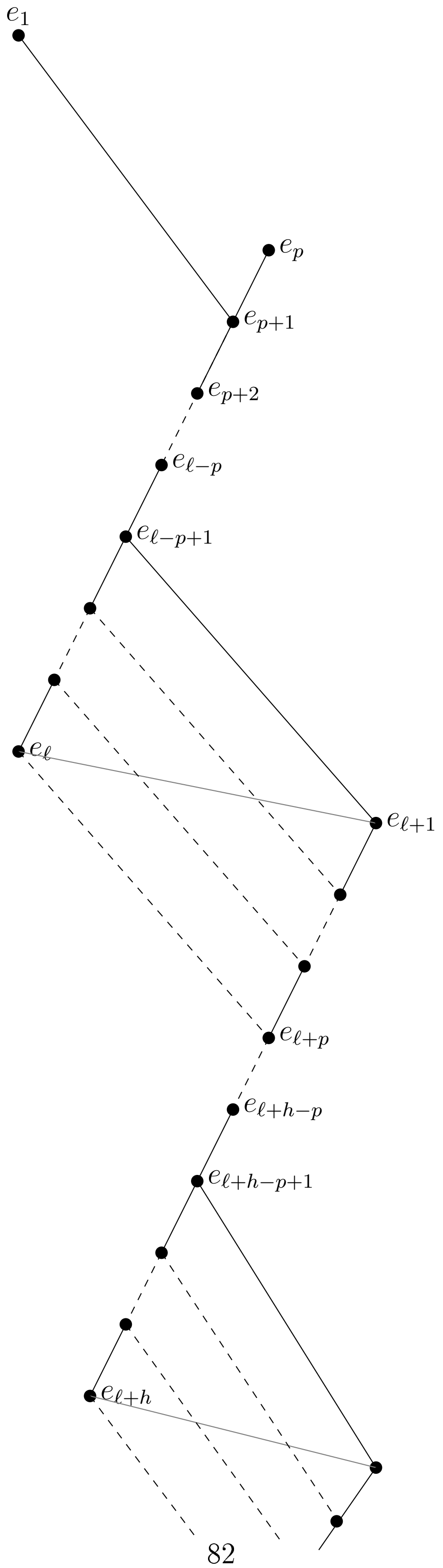

Figure 5.1: A generic Lie algebra of type p.

of type p. The relation between its two-step centralisers β_i and the two-step centralisers α_i of N is

$$[e_i, e_p] = [e_i, [y, e_1^{p-1}]] = \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} [e_{i+k}, y, e_1^{p-1-k}]$$
$$= \left(\sum_{k=0}^{p-1} \alpha_{i+k} \right) e_{i+p},$$

where the last equality comes from the fact that $\binom{p-1}{k} \equiv (-1)^k \mod p$, for $0 \leq k \leq p-1$. This means that $\beta_i = \sum_{k=0}^{p-1} \alpha_{i+k}$.

From the theory of Lie algebras of type 1 over fields of odd characteristic, we know that all two-step centralisers in a constituent coincide with $\alpha_2 = 0$ except for the last one. We also know that p is a lower bound for the length of all constituents ([CMN97]). We conclude that

- the constituents of L have exactly the same lengths of the constituents of N.

- the constituents of L have the form

$$(0, \ldots, 0, \underbrace{\lambda, \ldots, \lambda}_{p}),$$

  for some non-zero coefficient λ. Constituents of this form are called *ordinary ending in* λ.

We stress that the converse is also true: if a Lie algebra of maximal class of type p has ordinary ending constituents, then it is a subalgebra of a Lie algebra of type 1. We refer to [Ugo10, Lemma 3.1.2] for the proof of a more general fact. Figure 5.2 illustrates one of those Lie algebras. In this case, the lines at the end of every constituent are not dashed, meaning that the corresponding two-step centralisers are non-zero.

We present another example, which plays a very important role in the formulation of the classification theorem.

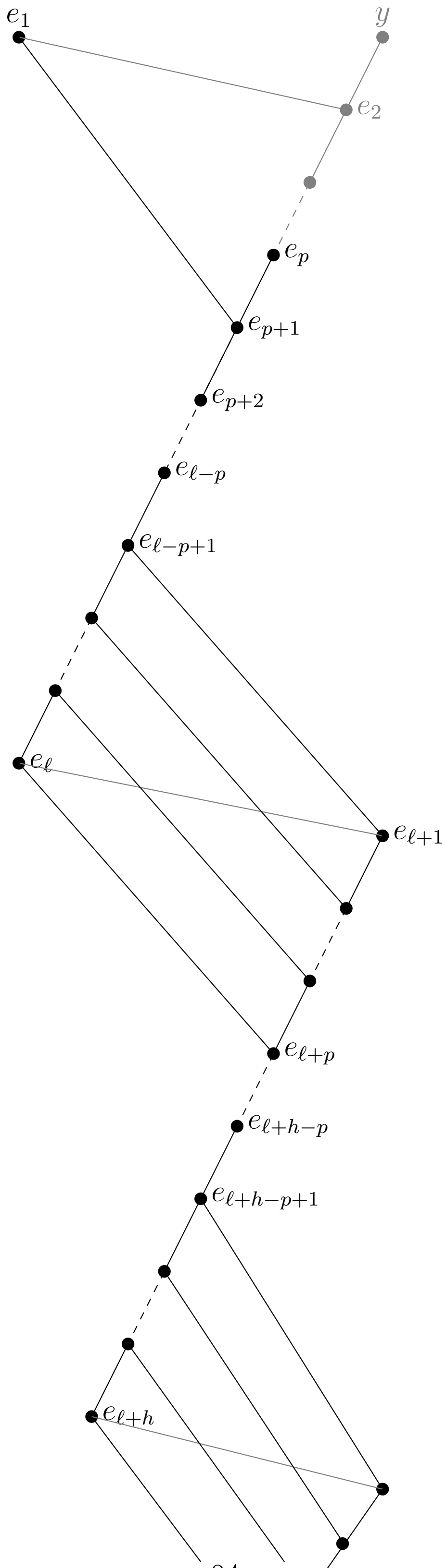

Figure 5.2: Subalgebra of type p of a Lie algebra of type 1.

Example 5.6 (**Translation of the two-step centralisers**). Suppose that L is a Lie algebra of maximal class of type p over a field F of characteristic p. Let e_1 and e_p be the generators of L and let $\{\beta_i\}_{i>p}$ be the corresponding sequence of two-step centralisers. We can translate it by a fixed constant $\delta \in F^*$, meaning that we replace $\{\beta_i\}_{i>p}$ by $\{\beta_i + \delta\}_{i>p}$.

This procedure has a clear interpretation if we embed L into an associative algebra A. For the rest of this example only, powers of an element refers to powers in such an associative algebra. In the rest of this chapter, we will use powers to denote repeated commutators. Taking $\delta \in F^*$, we consider the Lie subalgebra $L(\delta)$ of A, generated by e_1 and $e'_p = e_p + \delta e_1^p$. Then we have $e'_{p+1} = [e'_p, e_1] = e_{p+1}$ and more generally $e'_i = e_i$ for all $i > p$. Furthermore

$$\begin{aligned}[e_i, e'_p] &= [e_i, e_p + \delta e_1^p] = [e_i, e_p] + \delta[e_i, e_1^p] \\ &= \beta_i e_{i+p} + \delta[e_i, e_1^p] = (\beta_i + \delta)e_{i+p}.\end{aligned}$$

Here we have used the fact that in characteristic p the element of the basis $e_{i+p} = [e_p, \underbrace{e_1, \cdots, e_1}_{i}]$ equals $[e_i, e_1^p]$. It follows that $L(\delta) = L_1 \oplus \langle e'_p \rangle \oplus \bigoplus_{i>p} L_i$ is a Lie algebra of maximal class of type p with sequence of two-step centralisers $\{\beta_i + \delta\}_{i>p}$.

We summarise the conclusion of Example 5.6 as follows.

Lemma 5.7. *Let L be a graded Lie algebra of maximal class of type p over a field F of characteristic p, with sequence of two-step centralisers $\{\beta_i\}_{i>p}$. Then for any $\delta \in F$ there exists a Lie algebra of maximal class of type p with sequence of two-step centralisers $\{\beta_i + \delta\}_{i>p}$.*

Note that if the sequence of two-step centralisers of L is not constant and different from zero, then L and $L(\delta)$ are not isomorphic for $\delta \neq 0$. In fact, L and $L(\delta)$ are isomorphic precisely when there exists $\lambda \in F^*$ such that $\lambda\beta_i = \beta_i + \delta$, meaning that $\beta_i = \delta/(\lambda - 1)$ for all $i > p$ and L has constant sequence of two-step centralisers, all equal to 1 by the chosen normalisation.

Conversely, if $\beta_i \equiv \gamma$ for all $i > p$, then $\lambda = 1 + \frac{\delta}{\gamma}$ satisfies the equation above. Furthermore, note that this construction changes the constituents length.

We conclude this section by recalling Lucas' theorem.

Theorem 5.8 (Lucas)**.** *Let a and b be two non-negative integers and let p be a prime number. If the p-adic expansions of a and b are respectively $a = a_0 + a_1 p + \cdots + a_n p^n$ and $b = b_0 + b_1 p + \cdots + b_n p^n$, then*

$$\binom{a}{b} = \prod_{i=0}^{n} \binom{a_i}{b_i} \mod p.$$

Here we use the convention that $\binom{a_i}{b_i} = 0$ if $a_i < b_i$.

Lucas' theorem is the main tool for the evaluation of a binomial coefficient modulo a prime number and it will be widely used in the next sections.

5.3 Polynomials with some zero coefficients

In this section we present some propositions about polynomials. These results, that are interesting in their own right, will be used in Section 5.5 in order to prove a proposition regarding the length of the first constituent of a Lie algebra of maximal class of type p.

For any polynomial g we use the notation $[x^j]g(x)$ to express the coefficient of x^j in $g(x)$. Let F denote a field of characteristic p. The following lemma gives information about the exponent k in case the polynomial $(x-1)^k(x-a) \in F[x]$ has half of the highest coefficients (excluding the leading one) being zero.

Lemma 5.9. *Let F be a field of characteristic p and let $x - a \in F[x]$. Suppose that, for a natural number $k > 1$, we have*

$$[x^j](x-1)^k(x-a) = 0 \quad \text{for } k/2 + 1 \leq j \leq k. \tag{5.1}$$

Then the couple (k, a) *is one of the following: either* $(2, -2)$ *or* $(3, -3)$ *or* $(q, 0)$ *or* $(q-1, 1)$ *or* $(2q-1, 1)$*, where* $q \geq p$ *is some power of* p*.*

Moreover, if the condition (5.1) *holds in the range* $(k+1)/2 \leq j \leq k$*, then the conclusions are the same except that the couples* $(3, -3)$ *and* $(2q-1, 1)$ *can be excluded.*

Proof. Note that $(2, -2)$ and $(3, -3)$ are genuine solutions because the polynomials $(x-1)^2(x+2) = x^3 - 3x + 2$ and $(x-1)^3(x+3) = x^4 - 6x^2 + 8x - 3$ satisfy the required condition. Observe that the former case is not exceptional if $p = 2$ and the latter is not exceptional if $p = 3$. Now assume that $k > 3$.

If $a = 0$ then it easily follows that k is a power of p, so assume that $a \neq 0$. The condition amounts to

$$\binom{k}{j-1} = -a\binom{k}{j} \quad \text{for } k/2 + 1 \leq j \leq k. \tag{5.2}$$

We stress the fact that this equality holds in the field F, hence in characteristic p. In particular, because $\binom{k}{k} = 1$ all binomial coefficients $\binom{k}{j}$ are non-zero modulo p for $k/2 \leq j \leq k$, and by symmetry actually for $0 \leq j \leq k$.

Lucas' theorem then implies that all p-adic digits of k equal $p-1$ except possibly the highest, and so $k = bq - 1$ where q is a power of p and $0 < b < p$. Indeed, suppose that $k = k_0 + k_1 p + \cdots + k_t p^t$ is the p-adic expansion of k, where $k_t \neq 0$ and, by contradiction, $k_i \neq p-1$ for some $i < t$. Taking for instance $j = k_0 + k_1 p + \cdots + (k_i + 1)p^i$, we would have

$$\binom{k}{j} \equiv \binom{k_0}{k_0} \cdots \binom{k_{i-1}}{k_{i-1}} \binom{k_i}{k_i + 1} \pmod{p} = 0.$$

The congruence $\binom{k}{j-1} = -a\binom{k}{j}$ combined with the general binomial identity $(k - j + 1)\binom{k}{j-1} = j\binom{k}{j}$ implies $(k+1)a = (a-1)j$. Taking $j = k$ and $j = k - 1$, which we can as $k > 3$, we get $a = 1$ and $k \equiv -1 \pmod{p}$.

If $k < p$ this yields the desired conclusion $k = p - 1$, so we may now assume $k = bq - 1 \geq p$. A recursive application of the congruence (5.2) with the value of a just found yields

$$\binom{bq-1}{j} \equiv (-1)^{bq-1-j} \pmod p \quad \text{for } (bq-1)/2 \le j \le bq-1. \tag{5.3}$$

If $b > 2$ this collides with

$$\binom{bq-1}{(b-1)q-1} = \binom{(b-1)q+q-1}{(b-2)q+q-1} \equiv \binom{b-1}{b-2} \equiv b-1 \pmod p,$$

which follows from Lucas' theorem. Hence $k = q-1$ or $2q-1$ as desired.

If we suppose now that the range in the hypothesis of this lemma is extended to $(k+1)/2 \le j \le k$, then we find by direct calculation that $(3,-3)$ cannot be a solution. Indeed, in this case both coefficients of degrees 2 and 3 vanish and these would yield to two incompatible solutions for a.

Still under the extended range, if we assume that $k = 2q-1$ and $a = 1$, then we have

$$(x-1)^{2q-1}(x-1) = (x-1)^{2q} = (x^q-1)^2 = x^{2q} - 2x^q + 1.$$

However $(k+1)/2 = q$ and the coefficient of x^q should be zero. Hence the case $(k,a) = (2q-1,1)$ must be excluded as well. $\square$

Observe that, in the above lemma, the second range of null coefficients is actually bigger than the first only when k is odd. We can now prove a very useful fact about polynomials.

Proposition 5.10. *Let F be a field of positive characteristic p and let $k > p+1$ be a natural number. Suppose that $g(x) \in F[x]$ is a monic polynomial of degree $p-1$ such that*

$$[x^j](x-1)^k g(x) = 0 \quad \textit{for } (k+p)/2 \le j < k. \tag{5.4}$$

Then, either $p+1 < k < 2p$ or $2p < k < 3p$ or $k = 3p+1$ or $k = 2q-p+1$, or $q-p+1 \le k \le q+p-1$, where $q > p$ is a power of p.

Moreover, if $k = 2q-p+1$ or $k = q-p+1$, then $g(x) = (x-1)^{p-1}$. Also, if $k = q+k_0$ for some $k_0 = 1, \dots, p-1$, then $x^{k_0} | g(x)$.

Proof. Observe that the lower bound for k is natural since otherwise the required hypothesis would not even make sense. Write $k = k'p + k_0$ with $0 \leq k_0 < p$. Write $g(x) = \sum_{i=0}^{p-1} g_i x^i$, hence $g_{p-1} = 1$.

For a polynomial $f(x) \in F[x]$ and an integer i denote by $S_i\bigl(f(x)\bigr)$ the polynomial obtained from $f(x)$ by discarding all terms where x appears with an exponent not congruent to i modulo p. In particular, $f(x) = \sum_{i=0}^{p-1} S_i\bigl(f(x)\bigr)$.

We deal with the simplest case $k_0 = 0$ first. In this case we have

$$(x-1)^k g(x) = (x^p - 1)^{k'} g(x),$$

and hence $S_i\bigl((x-1)^k g(x)\bigr) = (x^p-1)^{k'} g_i x^i$ for any i. Taking $i = p-1$ our hypothesis yields

$$[x^j](x-1)^{k'} = 0 \quad \text{for } k'/2 \leq j < k',$$

where the left inequality is because $p - 1 \geq p/2$. Note that, since we have assumed $k > p + 1$, we have that $k' > 1$ so that the interval is not empty. Hence the condition $\binom{k'}{j} = 0$ for all $k'/2 \leq j < k'$, and for $0 < j < k'$ by symmetry, implies that $k' \geq p$ is a power of p, and then so is k, say $k = q > p$. Clearly we cannot get any information on $g(x)$ in this case.

Now suppose that $k_0 = 1$. Then

$$(x-1)^k g(x) = (x^p - 1)^{k'} (x-1) g(x),$$

and hence $S_i\bigl((x-1)^k g(x)\bigr) = (x^p-1)^{k'}(g_{i-1} - g_i)x^i$ for $0 < i < p$, and $S_0\bigl((x-1)^k g(x)\bigr) = (x^p-1)^{k'}(x^p - g_0)$. Taking $i = 0$, our hypothesis gives

$$[x^j](x-1)^{k'}(x - g_0) = 0 \quad \text{for } k'/2 + 1 \leq j \leq k',$$

because these values of j satisfy $(k'p+1+p)/2 \leq jp < k'p+1$. Since we have assumed $k > p+1$, we have that $k' > 1$. According to Lemma 5.9, we have one of the following possibilities: (k', g_0) equals either $(2, -2)$ or $(3, -3)$ or $(q', 0)$ or $(q'-1, 1)$ or $(2q'-1, 1)$ for some power $q' \geq p$ of p. Consequently,

k can assume one of the following values: $k = 2p+1$, $k = 3p+1$, $k = q+1$, $k = q-p+1$ or $k = 2q-p+1$ where $q = q'p > p$.

We get extra information when $g_0 = 1$, that is when $k = q-p+1$ or $k = 2q-p+1$. Taking now any $0 < i < p$, our hypothesis yields

$$[x^j](g_{i-1} - g_i)(x-1)^{k'} = 0 \quad \text{for } (k'+1)/2 \le j < k',$$

because these values of j satisfy $(k'p+1+p)/2 \le jp+i < k'p+1$. The lower inequality can be refined for specific values of i but we do not need to be precise here because it suffices to look at $j = k'-1$. For this value of j we find $g_i = g_{i-1}$, hence starting from $g_0 = 1$, we get that $g_i = 1$ for all i. In conclusion, we find that $g(x) = (x-1)^{p-1}$.

More generally suppose that $k = k'p + k_0$, where $1 < k_0 < p$. Then

$$(x-1)^k g(x) = (x^p - 1)^{k'} (x-1)^{k_0} g(x).$$

and hence

$$S_{k_0-1}((x-1)^k g(x)) = (x^p-1)^{k'} \big(x^p + \sum_{s=0}^{k_0-1} (-1)^{k_0-s} \binom{k_0}{s} g_{k_0-1-s}\big) x^{k_0-1}.$$

Our hypothesis gives

$$[x^j](x-1)^{k'} \big(x - \big(\sum_{s=0}^{k_0-1} (-1)^{k_0+1-s} \binom{k_0}{s} g_{k_0-1-s}\big)\big) = 0 \text{ for } (k'+1)/2 \le j \le k',$$

because these values of j satisfy $(k'p+k_0+p)/2 \le jp+k_0-1 < k'p+k_0$.

When $k' = 1$ this implies that

$$\sum_{s=0}^{k_0-1} (-1)^{k_0+1-s} \binom{k_0}{s} g_{k_0-1-s} = -1.$$

Otherwise, if $k' > 1$, then by applying Lemma 5.9, we get one of the following possibilities:

1. either $k' = 2$ and $\sum_{s=0}^{k_0-1} (-1)^{k_0+1-s} \binom{k_0}{s} g_{k_0-1-s} = -2$,

2. or $k' = q'$ and $\sum_{s=0}^{k_0-1}(-1)^{k_0+1-s}\binom{k_0}{s}g_{k_0-1-s} = 0$,

3. or $k' = q' - 1$ and $\sum_{s=0}^{k_0-1}(-1)^{k_0+1-s}\binom{k_0}{s}g_{k_0-1-s} = 1$,

where $q' \geq p$ is a power of p. Hence, either $k = p + k_0$ or $k = 2p + k_0$ or $k = q + k_0$ or $k = q - p + k_0$, where $q = q'p$ is a power of p as well.

Now we only need to prove that when $k = q + k_0$, for some power $q > p$ of p and $0 < k_0 < p$, we have $x^{k_0} | g(x)$. First observe that in this case

$$(x-1)^k g(x) = (x^q - 1)(x-1)^{k_0} g(x).$$

The condition expressed in the hypothesis holds in particular in the range $q \leq j \leq q + k_0 - 1$, and implies that

$$[x^j](x-1)^{k_0} g(x) = 0, \qquad \text{for all } 0 \leq j < k_0.$$

The conclusion follows observing that x^{k_0} and $(x-1)^{k_0}$ are coprime. $\square$

This proposition holds for every prime but it will be used in the next section, in case p is an odd prime, for proving a result about the first constituent of a Lie algebra of maximal class of type p over a field of characteristic p. In this application the parameter k equals $\ell - p + 1$, where ℓ is the length of the first constituent.

The cases $k = 2q - p + 1$ and $q - p + 1 < k \leq q + 1$ really occur, if we take k to be even. In particular, the Lie algebras for $k = 2q - p + 1$ and $k = q + 1$ come from uncovered Lie algebras of type 1. The others, instead, are explicitly constructed in Scarbolo's thesis as members of an exceptional family. Lie algebras where $q + 2 < k < q + p - 1$ do not occur. This will be shown in the last section. The cases $p + 1 < k < 2p$, $2p < k < 3p$ and $k = 3p + 1$ remain still open, as Scarbolo's thesis is inconclusive in these cases.

5.4 The length of the first constituent

In the classification of Lie algebras of maximal class of type p, a very crucial role is played by the first constituent. For instance its length, when sufficiently large, allows us to distinguish two big families. On one side there are Lie algebras which come from Lie algebras of maximal class of type 1. They can simply be their subalgebras (see Example 5.5) or they can be constructed from them by adding a constant term δ to every two-step centraliser, as described in Example 5.6. In both cases the first constituent has length divisible by p. In the remaining cases, instead, the Lie algebra is built up from an "exceptional" Lie algebra belonging to a finite family constructed by Scarbolo in his thesis.

Hence, the following proposition can be seen as a preliminary step towards the classification theorem. It restricts the possible values for the length of the first constituent.

Proposition 5.11. *Suppose that a graded Lie algebra of maximal class of type p over a field of odd characteristic p has its first constituent of length $\ell > 4p$. Then*

1. *either $\ell = 2q$, where q is a power of p,*
2. *or $\ell = q + j$, where q is a power of p and j is an odd integer such that $1 \leq j \leq p$.*

As already mentioned above, in case 2 the case where $j = p$ is substantially different from all the others. Nevertheless, we will not see the details of this fact here.

Proposition 5.11 virtually corresponds to [Sca14, Proposition 3.2], where the author omits the hypothesis $\ell > 4p$. The fact that we can assume $\ell > 2p$, since this hypothesis is not restrictive for the purpose of the classification, has been already highlighted by Scarbolo. However, we currently do not know if Lie algebras of maximal class of type p with first constituents of length $\ell = 2p + j$ or $\ell = 3p + j$ with $0 < j \le p$ do exist.

This section and the next one describe our contribution to the proof of Proposition 5.11. It consists in a new approach which has the advantage of reducing the calculations, making clearer those which remain. Indeed, almost all of the information is recovered from a small number of relations. The results about polynomials proved in the previous section will be used.

We start giving the relations satisfied by a Lie algebra of type p. Let F be a field of odd characteristic p. Suppose that L is a graded Lie algebra of maximal class of type p over F, generated by e_1 and e_p of degree respectively 1 and p. Hence for all $i > p$ we have $L_i = Fe_i$, where $e_i = [e_p, e_1^{i-p}]$. Suppose that its first constituent has length $\ell > 2p$ and the second one has length h. The sequence of its two-step centralisers is denoted by $\{\beta_i\}_{i>p}$. The following relations hold.

1. $[e_i, e_p] = 0$ for all $p \le i < \ell - p + 1$;
2. $[e_{\ell-p+1}, e_p] = [e_p, e_1^{\ell-2p+1}, e_p] \neq 0$;
3. $[e_{\ell-p+1}, e_1^p] = [e_p, e_1^{\ell-p+1}] = [e_{\ell-p+1}, e_p]$;
4. $[e_{\ell+j}, e_p] = 0$ for all $1 \le j \le h-p$, or equivalently $[e_{\ell-p+1}, e_p, e_1^i, e_p] = 0$ for all $0 \le i < h-p$;
5. $[e_{\ell+h-p+1}, e_p] \neq 0$, or equivalently $[e_{\ell-p+1}, e_p, e_1^{h-p}, e_p] \neq 0$.

The relations 1 and 2 define the first constituent. Identity 3 follows from 2 and the definition of two-step centralisers with the chosen normalisation. The relations 4 and 5 define the second constituent, using relation 3 as well. Observe that the assumption $\ell > 2p$ is equivalent to the condition $\beta_{p+1} = 0$. It is not restrictive for the purpose of the classification, essentially because we could possibly operate a translation of the two-step centralisers to fall into this case. The details of this fact can be found in [Sca14]. The next lemma gives us some information about the parity of the length ℓ of the first constituent.

Lemma 5.12. *Suppose that L is a graded Lie algebra of maximal class of type p over a field of odd characteristic p. Let ℓ denote the length of its first constituent. Then ℓ is even.*

Proof. Suppose by contradiction that this is not true. Then $\ell - 2p + 1$ is even and we can write

$$0 = [[e_p, e_1^{(\ell-2p+1)/2}], [e_p, e_1^{(\ell-2p+1)/2}]] = (-1)^{(\ell-2p+1)/2}[e_p, e_1^{\ell-2p+1}, e_p] \neq 0,$$

where we have used both the Jacobi identity and relations 1 and 2. The conclusion follows. □

We can also say something about the length h of the second constituent. The next result is a special case of [Sca14, Lemma 3.1].

Lemma 5.13. *Suppose that L is a graded Lie algebra of maximal class of type p over a field of odd characteristic p. Let ℓ denote the length of its first constituent, and h the length of its second constituent. Then $h \geq \ell/2$.*

Proof. Following the relations above, we need to prove that $[e_{\ell+j}, e_p] = 0$ for all $1 \leq j \leq (\ell/2) - p$. We will proceed by induction, the base case for $j = 1$ being already proved.

First of all, we know that $[[e_p, e_1^i], [e_p, e_1^{i+1}]] = 0$ for all $0 \leq i < (\ell/2) - p$, by Jacoby identity and relation 1. Now assume by inductive hypothesis that $[e_{\ell+j}, e_p] = 0$ for all $j = 1, \ldots, r$, where $r < (\ell/2) - p$. Then

$$\begin{aligned}
0 &= [e_{\ell-p-r}, [[e_p, e_1^r], [e_p, e_1^{r+1}]]] \\
&= -[e_{\ell-p-r}, [e_p, e_1^{r+1}], [e_p, e_1^r]] \\
&= (-1)^r \beta_{\ell-p+1}[e_{\ell+1}, [e_p, e_1^r]] \\
&= \beta_{\ell-p+1}[e_{\ell+r+1}, e_p]
\end{aligned}$$

and the conclusion follows. □

This lemma conclude the section of the preliminary results, necessary to the proof of Proposition 5.11.

5.5 Proof of Proposition 5.11

In this section, we prove Proposition 5.11 by applying the results in the previous two sections. We will first use all relations listed in the previous section to show that the last p two-step centralisers of the first constituent satisfy a linear recurrence. This can be interpreted as a condition on the coefficients of a certain polynomial which will allow the application of Proposition 5.10. The final part of the proof will then consist of excluding some of the possibilities obtained from the application of this proposition.

The relations 1-5 listed above give us information about the adjoint action of $e_{\ell+1}$. In particular, $e_{\ell+1}$ centralises e_{p+j-1} for $1 \leq j \leq h-p$. Indeed, we can write

$$\begin{aligned} [e_{\ell+1}, e_{p+j-1}] &= [[e_p, e_1^{\ell-p+1}], [e_p, e_1^{j-1}]] = [[e_{\ell-p+1}, e_p], [e_p, e_1^{j-1}]] \\ &= \left(\sum_{t=0}^{j-1} (-1)^t \binom{j-1}{t} [e_{\ell-p+1}, e_p, e_1^t, e_p, e_1^{j-1-t}] \right) = 0, \end{aligned}$$

since $t \leq j-1 \leq h-p-1$. Therefore, we have

$$0 = [[e_p, e_1^{j-1}], [e_p, e_1^{\ell-p+1}]] = \sum_{t=0}^{\ell-p+1} (-1)^t \binom{\ell-p+1}{t} [e_p, e_1^{j-1+t}, e_p, e_1^{\ell-p+1-t}].$$

Note that $j-1 \leq j-1+t \leq \ell-p+j \leq \ell+h-2p$. Hence, all Lie brackets involved vanish except possibly those for $j-1+t = \ell-2p+1, \ldots, \ell-p$, that is for $t = \ell-p+1-j-(p-1), \ldots, \ell-p+1-j$. We conclude

$$0 = \sum_{t=0}^{p-1} (-1)^{\ell-p+1-(j+t)} \binom{\ell-p+1}{\ell-p+1-(j+t)} [e_p, e_1^{\ell-p-t}, e_p, e_1^{j+t}]$$

from which, by symmetry of the binomial coefficients, we get that

$$\sum_{t=0}^{p-1} (-1)^{j+t} \binom{\ell-p+1}{j+t} \beta_{\ell-t} = 0, \qquad \text{for } 1 \leq j \leq h-p.$$

It is not hard to see that this is equivalent to

$$[x^j](x-1)^k\beta(x) = 0 \qquad \text{for } k-h+p-1 < j < k, \tag{5.5}$$

where $k = \ell-p+1$ and $\beta(x) = \beta_\ell+\beta_{\ell-1}x+\cdots+\beta_{\ell-p+1}x^{p-1} \in F[x]$ is a monic polynomial of degree $p-1$ with coefficients in the field F of characteristic p. Remember that the polynomial $\beta(x)$ can be assumed to be monic because of the chosen normalisation for the sequence of two-step centralisers. By using the fact that $h \geq \ell/2$, we get the condition

$$[x^j](x-1)^k\beta(x) = 0 \quad \text{for } (k+p-1)/2 < j < k.$$

This is equivalent to the hypothesis of Proposition 5.10. However, recall that k is even here. Because $\ell > 4p$ implies $k > 3p+1$, Proposition 5.10 gives us only the following possibilities: either $k = 2q-p+1$ or $q-p+1 \leq k \leq q+p-1$, where $q > p$ is a power of p. Equivalently, the possibilities for ℓ are either $\ell = 2q$ or $q < \ell \leq q+2p-1$, considering also the parity of ℓ. This is the end of our original contribution. The remaining part is a reinterpretation of Scarbolo's results.

In order to prove Proposition 5.11, we are left with excluding the cases $q+p < \ell < q+2p-1$, which correspond to $q+1 < k < q+p$. We proceed by contradiction, by assuming that $k = q+k_0$, where k_0 is an odd (by the parity of ℓ in Lemma 5.12) integer such that $1 < k_0 < p-1$. The length ℓ of the first constituent then equals $q+p+k_0-1$. Proposition 5.10 gives further information. Indeed, since g_i in that proposition corresponds to $\beta_{\ell-i}$, we have $\beta_{q+k_0+p-1} = \cdots = \beta_{q+p} = 0$. Note that this means that the last k_0 two-step centralisers of the first constituent vanish. We will use this fact to get the contradictory conclusion that $\beta_{q+p-r} = 0$ for all $0 \leq r \leq p-k_0$. Indeed, this fact is impossible as we know that $\beta_k = \beta_{q+k_0} = 1$.

In the next lemma we give an upper bound for the length h of the second constituent in the cases where $k = q+k_0$. The proof of this result can be found at the end of this section.

Lemma 5.14. *Let L be a graded Lie algebra of maximal class of type p over a field of odd characteristic p. Let ℓ be the length of its first constituent (recall that ℓ is even) and $k = \ell - p + 1 = q + k_0$, where $1 < k_0 < p$ and $q > p$ a power of p. Let h be the length of its second constituent. Then $h \leq q - 2$.*

We stress that the situation described in Lemma 5.14 will never happen. We said we know that

$$\beta_{q+k_0+p-1} = \cdots = \beta_{q+p} = 0$$

and we want to prove that $\beta_{q+p-r} = 0$ for all $r = 0, \ldots, p - k_0$, obtaining a contradiction from the last one. We proceed by induction, the base case being proved already. By inductive hypothesis suppose that the statement has been proved for all β_{q+p-j} where $j = 0, \ldots, r-1$. We want to prove it for $j = r$. Recall that the first non-zero two-step centraliser of the second constituent is β_{q+h+k_0}. We need to distinguish two cases according to the parity of r. Suppose that r is even and consider the equation

$$0 = [e_{h-p+k_0+r}, e_{q+p-r}] + [e_{q+p-r}, e_{h-p+k_0+r}].$$

By inductive hypothesis and the fact that $q+p-r+(h-2p+k_0+r) < q+h+k_0$, after expanding the second term in the usual way, we find that the only surviving term is for $j = 0$. Hence, we get

$$-\beta_{q+p-r}e_{q+h+k_0} = [e_{h-p+k_0+r}, e_{q+p-r}]. \tag{5.6}$$

Now consider the Jacobi identity

$$0 = -[e_{h-p+k_0+r}, [e_{q+p-r}, e_p]] + [e_{h-p+k_0+r}, e_{q+p-r}, e_p] - [e_{h-p+k_0+r}, e_p, e_{q+p-r}].$$

Note that the last term is zero since $h - p + k_0 + r < q + k_0$. By substituting Equation (5.6) into the second term we get

$$\begin{aligned}0 = -\beta_{q+p-r} \sum_{j=0}^{q+p-r} (-1)^j \binom{q+p-r}{j} [e_{h-p+k_0+r+j}, e_p, e_1^{q+p-r-j}] \\ - \beta_{q+p-r}[e_{q+h+k_0}, e_p].\end{aligned}$$

In the above formula all terms vanish except possibly those for $j = 0, \dots, p-r$ and $j = q, \dots, q + p - r$. In the first range $h - p + k_0 + r + j < q + k_0$, and consequently all Lie brackets involved vanish. In the latter case the only surviving Lie bracket is the last one. Then

$$0 = \beta_{q+p-r}[e_{q+h+k_0}, e_p](1 + (-1)^{q+p-r}),$$

from which, by the parity of r, we get $\beta_{q+p-r} = 0$.

Now suppose that r is odd and consider the equation

$$0 = [e_{h-p+k_0+r+1}, e_{q+p-r}] + [e_{q+p-r}, e_{h-p+k_0+r+1}].$$

Analogously to what we have seen above, we have

$$0 = [e_{h-p+k_0+r+1}, e_{q+p-r}] + \beta_{q+p-r} e_{q+h+k_0+1}, \tag{5.7}$$

and applying the adjoint action of e_p to both terms we get the corresponding Jacobi identity. Substituting Equation (5.7) to it we get

$$\begin{aligned} 0 = &-\beta_{q+p-r} \sum_{j=0}^{q+p-r} (-1)^j \binom{q+p-r}{j} [e_{h-p+k_0+r+1+j}, e_p, e_1^{q+p-r-j}] \\ &- \beta_{q+p-r}[e_{q+h+k_0+1}, e_p] \end{aligned}$$

All binomial coefficients are multiple of p except for the cases $j = 0, \dots, p - r$ and $j = q, \dots, q + p - r$. In the former case all Lie brackets vanish since $h - p + k_0 + r + 1 + j < q + k_0$. In the latter, the only surviving Lie brackets are obtained for the last two values of j. However, the last one equals $\beta_{q+p-r}[e_{q+h+k_0+1}, e_p]$, so after cancellation we are left with

$$0 = -\beta_{q+p-r}(q + p - r)[e_{q+h+k_0}, e_p, e_1]$$

from which we get $\beta_{q+p-r} = 0$, as desired. This concludes our induction and hence $\beta_{q+p-r} = 0$ for all $0 \leq r \leq p - k_0$. The contradiction against $\beta_{q+k_0} = 1$ implies that the cases $k = q+2, \dots, q+p-2$ are actually impossible.

Therefore, the length ℓ of the first constituent can only be either $\ell = 2q$ or $\ell = q+j$, where $q > p$ is a power of p and j is an odd number $1 \leq j \leq p$. This completes the proof of Proposition 5.11, aside from proving Lemma 5.14.

Proof of Lemma 5.14. Recall that $\beta(x)$ is the monic polynomial of degree $p-1$ with coefficients given by the last p two-step centralisers of the first constituent. We first show that the length h of the second constituent does not exceed q. In fact $(x-1)^k = (x^q - 1)(x-1)^{k_0}$ and we have

$$[x^{k_0+p-1}](x-1)^k \beta(x) = -[x^{k_0+p-1}](x-1)^{k_0}\beta(x) \neq 0.$$

Using this fact in Equation (5.5), we get that $k_0 + p - 1 \leq q + k_0 - h + p - 1$, from which the desired conclusion follows. It only remains to prove that $h \neq q, q-1$. Suppose by contradiction that $h = q$. Then the Lie algebra L satisfies

1. $[e_i, e_p] = 0$ for $p \leq i \leq q + k_0 - 1$,
2. $[e_{q+k_0}, e_p] \neq 0$,
3. $[e_{q+k_0+p-1+i}, e_p] = 0$ for $1 \leq i \leq q - p$,
4. $[e_{2q+k_0}, e_p] \neq 0$.

Consider the following Jacobi identity

$$0 = [e_{q+k_0-1}, e_{q+1}, e_p] - [e_{q+k_0-1}, e_p, e_{q+1}] + [e_{q+1}, e_p, e_{q+k_0-1}]. \tag{5.8}$$

The last two terms vanish because of 1. Expanding the first one, we get

$$0 = \sum_{j=0}^{q+1-p} (-1)^j \binom{q+1-p}{j} [e_{q+k_0-1+j}, e_p, e_1^{q+1-p-j}, e_p].$$

All binomial coefficients are multiple of p except when $j \not\equiv 0, 1 \pmod{p}$. Moreover, the Lie brackets involved vanish unless $j \leq p$. Since $\beta_{q+k_0-1} = 0$

by identity 1 and $\beta_{q+k_0+p-1} = 0$ for Proposition 5.10, only the term for $j = 1$ survives and gives

$$0 = [e_{q+k_0} e_p, e_1^{q-p}, e_p] = \beta_{q+k_0}[e_{2q+k_0}, e_p] = \beta_{q+k_0}\beta_{2q+k_0} e_{2q+p+k_0},$$

against relations 2 and 4. Hence $h \neq q$.

Now suppose that $h = q - 1$. This means that L satisfies

1. $[e_i, e_p] = 0$ for $p \leq i \leq q + k_0 - 1$,

2. $[e_{q+k_0}, e_p] \neq 0$,

3. $[e_{q+k_0+p-1+i}, e_p] = 0$ for $1 \leq i \leq q - p - 1$,

4. $[e_{2q+k_0-1}, e_p] \neq 0$.

Considering Equation (5.8), the same argument give the identity

$$\beta_{q+k_0}[e_{2q+k_0}, e_p] = 0,$$

which implies $[e_{2q+k_0}, e_p] = 0$. This can be used in the equation

$$0 = [e_{q+k_0+1}, e_p, e_{q-1}] - [e_{q+k_0+1}, e_{q-1}, e_p] - [e_{q-1}, e_p, e_{q+k_0+1}]$$

to show that the second term on the right-hand side vanishes. The third vanishes as well by relation 1. After expanding the first term in the usual way we find

$$0 = \beta_{q+k_0+1} \sum_{i=0}^{q-1-p} (-1)^i \binom{q-1-p}{i} [e_{q+p+k_0+1+i}, e_p, e_1^{q-1-p-i}].$$

Using the condition $[e_{2q+k_0} e_p] = 0$ again, we get that the only non-zero term is obtained for $j = q - 2 - p$. Consequently

$$0 = (q - p - 1)\beta_{q+k_0+1}[e_{2q+k_0-1}, e_p, e_1],$$

which implies $\beta_{q+k_0+1} = 0$, by relation 4. However, this is impossible since the identity

$$0 = [[e_p, e_1^{(q+k_0+1-p)/2}], [e_p, e_1^{(q+k_0+1-p)/2}]]$$

implies that $[e_{q+k_0+1}, e_p] = (q+k_0+1-p)/2[e_{q+k_0}, e_p]$ and both terms on the right-hand side are non-zero. We conclude that $h \leq q-2$. $\square$

Note that Proposition 5.10 provides further information. Indeed , if $\ell = 2q$ then the first constituent is ordinary ending in 1. Moreover, if $\ell = q+p$, then the last two-step centraliser of the first constituent β_ℓ is zero.

Bibliography

[Bak87] Y. Bakhturin, Identical relations in Lie algebras, VNU Science Press, Utrecht, 1987.

[Ber73] T. Berger, Nilpotent fixed point free automorphism groups of solvable groups, Math. Z. 131 (1973), 305–312.

[BH96] V. Belyaev and B. Hartley, Centralizers of finite nilpotent subgroups in locally finite groups, Algebra Logic 35 (1996), 217–228.

[Bur97] W. Burnside, Theory of Groups of Finite Order, Cambridge Univ. Press, 1897.

[CMN97] A. Caranti, S. Mattarei, and M. Newman, Graded Lie algebras of maximal class, Trans. Amer. Math. Soc. 349 (1997), 4021–4051.

[CN00] A. Caranti and M. Newman, Graded Lie algebras of maximal class II, J. Algebra 229 (2000), 750 – 784.

[Col90] M. Collins, Some infinite Frobenius groups, J. Algebra 131 (1990), 161 – 165.

[CVL00] A. Caranti and M. Vaughan-Lee, Graded Lie algebras of maximal class IV, Ann. Sc. Norm. Super. Pisa CI. Sci. 29 (2000), 269–312.

[CVL03] ———, Graded Lie algebras of maximal class V, Israel J. Math. 133 (2003), 157–175.

[DM96] J. Dixon and B. Mortimer, Permutation groups, Springer, 1996.

[EG18] G. Ercan and İ. Güloğlu, Frobenius groups of automorphisms with almost fixed point free kernel, J. Algebra 521 (2018).

[EGK14a] G. Ercan, İ. Güloğlu, and E. Khukhro, Frobenius-like groups as groups of automorphisms, Turk. J. Math. 38 (2014), 965–976.

[EGK14b] ______, Rank and order of a finite group admitting a Frobenius-like group of automorphisms, Algebra Logic 53 (2014), 258–265.

[Gor68] D. Gorenstein, Finite groups, Harper and Row, 1968.

[Hal58] P. Hall, Some sufficient conditions for a group to be nilpotent, Illinois J. Math. 2 (1958), 787–801.

[HB82] B. Huppert and N. Blackburn, Finite groups II, Springer-Verlag, 1982.

[Hig57] G. Higman, Groups and rings having automorphisms without non-trivial fixed elements, J. London Math. Soc. (1957), 321–334.

[HJ60] M. Hall Jr, The theory of groups, Macmillan, New York, 1960.

[Hug85] I. Hughes, Groups with fixed-point-free automorphisms, C. R. Math. Rep. Acad. Sci. Canada 7 (1985), 61–66.

[Hup67] B. Huppert, Endliche gruppen I, Springer, 1967.

[Hup98] ______, Character theory of finite groups, W. De Gruyter, 1998.

[Isa76] M. Isaacs, Character theory of finite groups, Academic Press, 1976.

[Isa08] ______, Finite Group Theory, AMS, 2008.

[Jur05] G. Jurman, Graded Lie algebras of maximal class III, J. Algebra 284 (2005), 435–461.

[Khua] E. Khukhro, Finite groups admitting a Frobenius group of automorphisms with fixed-point-free kernel, In: Fourth Group Theory Conference of Iran, 7-9 March 2012, Isfahan, Iran.

[Khub] ______, Linear methods in the study of automorphisms, In: Algorithmic problems in Group Theory and related areas, July, 30 - August, 10, 2012, Chemal, Altai, Russia.

[Khu92] ______, Groups and Lie rings admitting an almost regular automorphism of prime order, Math. USSR-Sb. 71 (1992), 51–63.

[Khu93] ______, Nilpotent groups and their automorphisms, W. De Gruyter, 1993.

[Khu98] E. Khukhro, p-Automorphisms of Finite p-Groups, Cambridge Univ. Press, 1998.

[Khu02] E. Khukhro, Finite groups of bounded rank with an almost regular automorphism of prime order, Sib. Math. J. 43 (2002), 955–962.

[Khu08] ______, Graded Lie rings with many commuting components and an application to 2-Frobenius groups, Bull. Lond. Math. Soc. 40 (2008), 907–912.

[Khu10] ______, Fixed points of the complements of Frobenius groups of automorphisms, Sib. Math. J. 51 (2010), 552–556.

[Khu12] E. Khukhro, Fitting height of a finite group with a Frobenius group of automorphisms, J. Algebra 366 (2012), 1 – 11.

[Khu13] E. Khukhro, Rank and order of a finite group admitting a Frobenius group of automorphisms, Algebra Logic 52 (2013), 72–78.

[KK63] V. Kreknin and A. Kostrikin, Lie algebras with regular automorphisms, Soviet Math. Dokl. 4 (1963), 355–358.

[KM13a] E. Khukhro and N. Makarenko, Finite groups and Lie rings with a metacyclic Frobenius group of automorphisms, J. Algebra 386 (2013), 77 – 104.

[KM13b] E. Khukhro and N. Makarenko, Lie algebras admitting a metacyclic Frobenius group of automorphisms, Sib. Math. J. 54 (2013), 99–113.

[KMS14] E. Khukhro, N. Makarenko, and P. Shumyatsky, Frobenius groups of automorphisms and their fixed points, Forum Math. 26 (2014), 73–112.

[Kre63] V. Kreknin, The solubility of Lie algebras with regular automorphisms of finite period, Soviet Math. Dokl. 4 (1963), 683–685.

[KS11] E. Khukhro and P. Shumyatsky, Nilpotency of finite groups with Frobenius groups of automorphisms, Monatsh. Math. 163 (2011), 461–470.

[LGN80] C. Leedham-Green and M. Newman, Space groups and groups of prime-power order I, Arch. Math. 35 (1980), 193–202.

[Mag50] W. Magnus, A connection between Baker-Hausdorff formula and a problem of Burnside, Ann. Math. 52 (1950), 111 – 126.

[Mak07] N. Makarenko, Graded Lie algebras with few nontrivial components, Sib. Math. J. 48 (2007), 95–111.

[MK10] V. Mazurov and E. Khukhro, Unsolved Problems in Group Theory. The Kourovka Notebook. no. 17, Institute of Mathematics, Novosibirsk, 2010.

[MS10] N. Makarenko and P. Shumyatsky, Frobenius groups as groups of automorphisms, Proc. Amer. Math. Soc. 138 (2010), 3425–3436.

[Pas68] D. Passman, Permutation groups, W. A. Benjamin, New York, 1968.

[Row95] P. Rowley, Finite groups admitting a fixed-point-free automorphism group, J. Algebra 174 (1995), 724–727.

[Sca14] C. Scarbolo, Graded Lie algebras of maximal class in characteristic p, generated by two elements of degree 1 and p, PhD Thesis, Università degli Studi di Trento, 2014.

[Sha93] A. Shalev, Automorphisms of finite groups of bounded rank, Israel J. Math. 82 (1993), 395–404.

[Sha00] ______, Lie methods in the theory of pro-p groups, New Horizons in pro-p Groups, Birkhäuser Boston, Oxford, 2000, pp. 1–54.

[Shu98] P. Shumyatsky, Applications of Lie ring methods to Group Theory, Non-associative algebra and its applications, CRC Press, 1998, pp. 373–395.

[Shu05] ______, On the nilpotency class of Lie rings with fixed-point-free automorphisms, Turk. J. Math. 29 (2005), 65 – 74.

[Shu11] P. Shumyatsky, On the exponent of a finite group with an automorphism group of order twelve, J. Algebra 331 (2011), 482 – 489.

[STW05] P. Shumyatsky, A. Tamarozzi, and L. Wilson, $\mathbb{Z}_n$-graded Lie rings, J. Algebra 283 (2005), 149–160.

[SZ97] A. Shalev and E. Zelmanov, Narrow Lie algebras: a coclass theory and a characterization of the Witt algebra, J. Algebra 189 (1997), 294 – 331.

[Tho59] J. Thompson, Finite groups with fixed-point-free automorphisms of prime order, Proc. Nat. Acad. Sci. 45 (1959), 578–581.

[Tur84] A. Turul, Fitting height of groups and of fixed points, J. Algebra 86 (1984), 555 – 566.

[Ugo10] S. Ugolini, Graded Lie algebras of maximal class in positive characteristic, generated by two elements of different weights, PhD Thesis, Università degli Studi di Trento, 2010.

[Ver70] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, Bull. Soc. Math. France 98 (1970), 81–116.

[VL90] M. Vaughan-Lee, The restricted Burnside problem, vol. 5, Clarendon, Oxford, 1990.

[Wil07] L. Wilson, The nilpotency class of groups with fixed point free automorphisms of prime order, Cambridge Univ. Press, 2007.

[WZ92] J. Wilson and E. Zelmanov, Identities for Lie algebras of pro-p groups, J. Pure Appl. Algebra 81 (1992), 103 – 109.

[Zel90] E. Zelmanov, On some problems of Group Theory and Lie algebras, Math. USSR-Sb. 66 (1990), 159–168.